Invariant fat distributions on compact homogeneous spaces

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Fat distributions

Given a vector subbundle (distribution) $D \subset TM$ of the tangent bundle of a smooth manifold M, its *Levi-Tanaka form* $\mathfrak{L}_D$ is the skew-symmetric $C^\infty(M)$-bilinear map

$$\mathfrak{L}_D : \Gamma D \times \Gamma D \to \Gamma(TM/D),$$

defined by

$$\mathfrak{L}_D(X,Y) := \pi[X,Y], \qquad \pi : TM \to TM/D \text{ canonical projection.}$$

A distribution D is called *fat* or *strongly bracket generating* provided

$$(\mathfrak{L}_D)_x(v,-) \text{ is onto for each } x \in M \text{ and every non zero } v \in D_x.$$

This notion has been widely studied in the context of sub-Riemannian geometry (cf. [5]).

Main result

Theorem *Let $M = G/H$ be a homogeneous space of a compact Lie group G. If M admits a G-invariant fat distribution, then $\pi_1(M)$ is finite.*

An application to CR geometry

Corollary *Let (M, HM, J) be a connected, compact homogeneous CR manifold of hypersurface type with nondegenerate Levi form. Then $\pi_1(M)$ is finite.*

This was proved in the strongly pseudo convex case by H. Burns and S. Schnider in 1976 (see [2]) and in the general case in 1985 by H. Azad, A. Huckleberry and W. Richthofer [2]. Our proof is more elementary.

To apply our result in this case, it suffices to use the fact that M always admits a *compact* transitive group G of CR automorphisms (a direct proof of this fact has been provided by A. Spiro in [4]).

Sketch of the proof

Fix a bi-invariant metric on G and consider the induced G-invariant normal metric g on G/H. Let $o = \pi(e)$, $p : G \to G/H$ being the canonical projection.

Claim: *The existence of a G–invariant fat distribution forces the Ricci tensor of g to be positive definite, so that Myers' Theorem applies.*

- Expression of the Ricci tensor of g at o on $T_oM \cong \mathfrak{h}^\perp =: \mathfrak{m}$, where $\mathfrak{h} = Lie(H)$:

$$Ric_o(X,X) = \frac{1}{4}\sum_{i=1}^{n} ||[X,E_i]_{\mathfrak{m}}||^2 + \sum_{i=1}^{n} ||[X,E_i]_{\mathfrak{h}}||^2 = \frac{1}{4}\sum_{i=1}^{n} ||T_o(X,E_i)||^2 + \sum_{i=1}^{n} ||[X,E_i]_{\mathfrak{h}}||^2.$$

  Here $n = \dim(M)$, $\{E_i\}$ is an orthonormal basis of $\mathfrak{m}$ and T is the torsion of the canonical G-invariant linear connection.

  Hence it suffices to show that the null space $\mathfrak{z}$ of $T_o : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is trivial.

- We observe that the Levi form $\mathfrak{L} : \Gamma D \times \Gamma D \to \Gamma D^\perp$ can be expressed by means of T as:

$$\mathfrak{L}(X,Y) = -PT(X,Y) \qquad X, Y \in \Gamma D \tag{1}$$

  where $P : TM \to D^\perp$ is the orthogonal projection with respect to g.

- Using (1) and the fact that T is totally skew symmetric, one checks that fatness forces $\mathfrak{z} = \{0\}$.

References

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