Bergman kernels in several complex variables and matrix Schrödinger operators

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In this talk I will give an overview of some results I obtained relating pointwise estimates of Bergman kernels in several complex variables and a generalized class of Schrödinger operators with matrix-valued potentials.

Given a domain $\Omega \subseteq \mathbb{C}^{n+1}$, it is a classical problem to study the associated Bergman kernel, i.e., the integral kernel $B_\Omega(z,w)$ of the orthogonal projector of $L^2(\Omega)$ onto the subspace consisting of holomorphic functions. Deep results of Hörmander, Kohn, and Catlin among others suggested that good estimates should hold for pseudoconvex domains satisfying a finite-type condition. A fruitful way to approach the problem of setablishing these bounds is to study the *model domains*

$$\Omega = \{(z', z_{n+1}) \in \mathbb{C}^{n+1} :\ \Im(z_{n+1}) > \varphi(z')\}, \tag{1}$$

where φ is a plurisubharmonic non-harmonic real-valued polynomial of n complex variables. A reduction to the boundary and a Fourier transform in the real part of z_{n+1} allow to reduce the problem to the study of the orthogonal projector of

$$L^2(\Omega, \varphi) = \left\{ f : \mathbb{C}^n \to \mathbb{C} :\ \int_{\mathbb{C}^n} |f|^2 e^{-2\varphi} < +\infty \right\} \tag{2}$$

onto the subspace consisting of holomorphic functions. The integral kernel $K_\varphi(z,w)$ of this operator is called the *weighted Bergman kernel* (corresponding to φ). Using a technique invented by Kerzman, one can relate the values of $K_\varphi(z,w)$ to a second order elliptic operator $\Box_\varphi$ called the *weighted Kohn Laplacian* acting on $(0,1)$-forms on $\mathbb{C}^n$. It was observed by Berndtsson that when $n = 1$ the weighted Kohn Laplacian is unitarily equivalent to a magnetic Schrödinger operator, and Christ used that observation to prove pointwise bounds for $K_\varphi(z,w)$ in one complex variable.

We observe that when $n \geq 2$ the weighted Kohn Laplacian is unitarily equivalent to a generalized Schrödinger operator whose electrical potential is matrix-valued. Inspired by this fact, we develop a version of Agmon theory (originally introduced to prove exponential decay of eigenfunctions of Schrödinger operators) for weighted Kohn Laplacians, and use that to deduce new pointwise bounds for $K_\varphi(z,w)$ from coercivity bounds of the form

$$\Box_\varphi \geq \mu^2, \tag{3}$$

where $\mu : \mathbb{C}^n \to \mathbb{R}$ and the inequality is in the sense of self-adjoint operators. The larger μ, the better the bound.

It is therefore interesting to establish (3) when φ is a plurisubharmonic non-harmonic polynomial with μ as large as possible. This appears to be a non-trivial problem, and we studied it carefully when $n = 2$ and the weight is of the form

$$\varphi(z_1, z_2) = \sum_{(\alpha_1,\alpha_2)\in\Gamma} |z_1^{\alpha_1} z_2^{\alpha_2}|^2, \tag{4}$$

where $\Gamma \subseteq \mathbb{N}^2$ is finite. We proved that (3) holds for these weights with $\mu(z_1, z_2) = 1 + |z_1|^{a_1} + |z_2|^{a_2}$, where $a_1, a_2 \geq 0$ are easily computable from Γ. Two ingredients used in the proof are linear programming arguments and a new *holomorphic uncertainty principle* inspired by the unitary equivalence of $\Box_\varphi$ and matrix Schrödinger operators.

References

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