Minimal unit vector fields with respect to Riemannian natural metrics

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1 Introduction

Let (M, g) be a compact n-dimensional Riemannian manifold. We denote by G_S the Sasaki metric on the tangent bundle TM and by $\tilde{G}_S$ the Sasaki metric induced by G_S on the unit tangent sphere bundle T_1M. Denote by $\mathfrak{X}^1(M)$ the set of all unit smooth vector fields on M which we suppose to be non-empty (this implies that the Euler-Poincaré characteristic of M vanishes). Then we can define the volume of a unit vector field $V \in \mathfrak{X}^1(M)$ as the volume of the submanifold $V(M)$ of the unit tangent sphere bundle T_1M equipped with Sasaki metric $\tilde{G}_S$. In a paper published in 1986 [14], H. Gluck and W. Ziller set the problem to find the unit vector fields of minimum volume, and settled the question for the unit *three*-sphere $\mathbb{S}^3$: *the unit vector fields of minimum volume on* $\mathbb{S}^3$ *are the Hopf vector fields and no others.* The unit tangent sphere bundle $T_1\mathbb{S}^3$, which can be identified with the Stiefel manifold $V_2\mathbb{R}^4$, can be equipped with the metric $\tilde{G}_0$ induced by the Euclidean metric of $\mathbb{R}^8$. H. Gluck and W. Ziller proved their result by using the method of calibrated geometries of Federer and Harvey-Lawson, and remarked ([14],p.180) that such result is true regardless of which of the two natural Riemannian metrics $\tilde{G}_S$ and $\tilde{G}_0$ is used to measure the volume of vector fields.

For a given M, unit vector fields of minimum volume, if they exist, are to be found among the critical points of the volume functional restricted to $\mathfrak{X}^1(M)$. Then, O. Gil-Medrano and E. Llinares-Fuster [13] computed the first variation of the volume functional $F : \mathfrak{X}^1(M) \to \mathbb{R}$, $V \mapsto F(V) = \mathrm{vol}(M, V^*\tilde{G}_S)$. The critical points of such functional F are called *minimal unit vector fields.* It is worthwhile to note that minimal vector fields correspond to minimal submanifolds, i.e., minimal vector fields are also critical points for the volume functional defined on the larger space of all immersion of M into T_1M [13]. In the last fifteen years, an important number of papers have appeared containing theoretical results and examples on minimality of vector fields in different geometrical situations, mainly with respect to the Sasaki metric. We refer [8],[9],[12],[20]-[22],[24],[25], [27],[28] and the references therein for an (incomplete) survey of the present state of the research in this field.

In a paper published in 1988 [18], O. Kowalski and M. Sekizawa have completely classified what are now called *Riemannian g-natural metrics*, o more simply *Riemannian natural metrics*, on TM. For the idea and the concept of naturality see the basic monograph by I. Kolář, P.W. Michor and J. Slovák [17]. We refer to [1],[2],[17],[18] for more details about this topic. The family of Riemannian natural metrics G on TM, which includes the Sasaki metric G_S and other well known Riemannian metrics on TM, is very large, it depends on six arbitrary smooth real functions [2],[18]. The restrictions $\tilde{G}$ of such metrics to the tangent sphere bundle T_1M possess a simpler form and globally depend on four real parameters, satisfying some inequalities. The Sasaki metric $\tilde{G}_S$, the *Euclidean metric* $\tilde{G}_0$, the *Cheeger-Gromoll metric* and the *Kaluza-Klein metrics* (as commonly defined on principal bundles [15]) are examples of Riemannian natural metrics $\tilde{G}$ on T_1M. Recently, Riemannian g-natural metrics on TM and T_1M have been intensively studied under different points of view. Some examples can be found in Refs. [3]-[5], [10], [11], [22], [23] and references therein. Some results about minimal unit vector fields with respect to some special Riemannian natural metric on T_1M are given in [22]. However, it was not known the critical point condition that defines minimal unit vector fields with respect to an arbitrary Riemannian g-natural metric $\tilde{G}$ (this problem was posed in [22], p.1201).

The main purpose of this paper is to investigate unit vector fields which define minimal unit vector fields with respect to an arbitrary Riemannian natural metric $\tilde{G}$. The paper is organized in the following way. We shall report in Section 2 some basic information on Riemannian natural metrics on tangent and unit tangent sphere bundles. In Section 3 we determine the first variational formula for the volume functional $F_{\tilde{G}} = \mathrm{vol}_{\mathfrak{X}^1(M)} : V \mapsto \mathrm{vol}(M, V^*\tilde{G})$ with respect to an arbitrary Riemannian natural metric $\tilde{G}$ on T_1M. Then, *a unit vector field V is a critical point of such functional if and only if a 1-form $\tilde{\omega}_V$, given in terms of derivate covariant of V, vanishes* (see Theorem 3.1). In Section 4, we define a minimal unit vector field as a critical point of the volume functional $F_{\tilde{G}}$. Since the critical point condition has a tensorial character, we assume this condition to define minimal unit vector fields (with respect to $\tilde{G}$) also in the non compact case. We note that, *the minimality condition with respect to the Sasaki metric $\tilde{G}_S$ is invariant under a two-parameters deformation of the Sasaki metric. In particular, a unit vector field is minimal with respect to a Kaluza-Klein metric if and only if it is minimal with respect to the Sasaki metric.*

Then, we show that *a unit vector field V defines a minimal immersion $V : M \to (T_1M, \tilde{G})$ if and only if V is a minimal unit vector field.* The last Section is devoted to give examples of minimal unit vector fields (with respect to $\tilde{G}$). Using a criteria of minimality for unit Killing vector fields, we get that *the Hopf vector fields of the unit sphere, the Reeb vector field of a K-contact manifold, and the Hopf vector field of a quasi-umbilical hypersurface with constant principal curvatures in a Kähler manifold, are minimal vector fields (with respect to $\tilde{G}$).* In particular the result of Johnson [16], about the minimality of Hopf vector fields on $\mathbb{S}^{2n+1}$, is invariant under a four-parameter deformation of the Sasaki metric $\tilde{G}_S$. Finally, we see that there exist examples of unit vector fields which are minimal with respect to a g-natural metric and not minimal with respect to another g-natural metric.

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