AN INVITATION TO THE DRURY-ARVESON SPACE

NICOLA ARCOZZI

ABSTRACT. This short invitation to the Drury-Arveson space and to some related open problems is a slightly expanded version of the talk given at the *Workshop su varietà reali e complesse* held at the SNS in Pisa, 28/2-3/3 2013.

The Drury-Arveson space is a Hilbert space of functions in several complex variables, playing in multivariable operator theory a role analogous to that of the Hardy space. Its reproducing kernel reflects the invariant geometry of the unit ball in a subtle way.

(i) The holomorphic Hardy space. The holomorphic Hardy space $H^2 = H^2(\Delta)$ on the unit disc Δ of the complex plane, owes much of its popularity to the pivotal role it plays at the interface between function theory and operator theory. This can be exemplified by some famous results. For instance, let me consider the von Neumann inequality. Let $A : H \to H$ be a linear contraction on a Hilbert space H and let $p(z)$ be a polynomial in the complex variable z. What is the best guess we can make on the norm $|||p(A)|||_{\mathcal{B}(H)}$? In 1951 von Neumann showed that

$$|||p(A)|||_{\mathcal{B}(H)} \leq \sup_{|z|<1} |p(z)| =: \|p\|_{H^\infty}, \tag{0.1}$$

with equality when $H = H^2$ and $A = M_z$ is the operator "multiplication times z". To better appreciate the result it is useful to know that $H^\infty = \mathcal{M}(H^2)$ isometrically: if $M_\varphi : f \mapsto \varphi f$ is the multiplication operator with holomorphic symbol φ, then

$$|||M_\varphi|||_{\mathcal{B}(H^2)} = \|\varphi\|_{H^\infty}.$$

The proof is elementary. The inequality $\geq$ follows from the general fact that H^2 is a Hilbert space with reproducing kernel. See the calculations in (vi) below. For the opposite inequality, one has to consider specific properties of H^2, in particular its norm. If $f(z) = \sum_{n=0}^\infty a_n z^n$, then

$$\begin{aligned}
\|f\|_{H^2}^2 &= \sum_{n=0}^\infty |a_n|^2 \\
&= \sup_{0\leq r<1} \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(re^{it})|^2 dt \\
&= \lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(re^{it})|^2 dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(e^{it})|^2 dt \\
&\quad \text{(it can be proved that } f \text{ has radial boundary limits } a.e.) \\
&= |f(0)|^2 + \frac{1}{\pi}\int_\Delta |f'(z)|^2 \log(1/|z|^2) dxdy \\
(*) \quad &\approx |f(0)|^2 + \int_\Delta |f'(z)|^2 (1-|z|^2) dxdy.
\end{aligned} \tag{0.2}$$

Boundedness of M_φ under the assumption that φ belongs to H^∞ easily follows from the fourth "avatar" of the H^2 norm, and almost as easily from the second and third. Note that the ℓ^2 version of the H^2 norm is not amenable to proving boundedness of M_b. Underlying this is the fact that, in order to characterize the L^2 boundedness of convolution operators, some version of the Fourier version is of great help: if $\varphi(z) = \sum_{n=0}^\infty b_n z^n$, then $M_\varphi f(z) = \sum_{n=0}^\infty (a*b)(n) z^n$, where

$$a * b(n) = \sum_{m=0}^\infty a_{n-m} b_m = \sum_{m=0}^n a_{n-m} b_m.$$

This is one the main reasons why electrical engineers, who interpret a as a positive-time, finite energy signal and b as a time-invariant, realizable, stable filter, are interested in H^2 and H^∞. The *wrong* choice $(*)$ of the norm leads to an interestingly inefficient proof of the inequality $|||M_\varphi|||_{\mathcal{B}(H^2)} \lesssim \|\varphi\|_{H^\infty}$:

$$\begin{aligned}\|M_\varphi f\|_{H_2}^2 &\approx |\varphi(0)f(0)|^2 + \int_\Delta |(\varphi f)'(z)|^2 (1-|z|^2) dxdy \\ &\lesssim \|\varphi\|_{H^\infty}^2 \left(|f(0)|^2 + \int_\Delta |f'(z)|^2(1-|z|^2)dxdy\right) \\ &\quad + \int_\Delta |f(z)|^2 d\mu_\varphi(z),\end{aligned}$$

with $d\mu_\varphi(z) = |\varphi'(z)|^2(1-|z|^2)dxdy$. Now, the requirement that the second summand in the last inequality is bounded by $\|f\|_{H^2}^2$ is usually expressed by saying that $d\mu_\varphi$ is a *Carleson measure* for the Hardy space. It was proved by Fefferman [12] that the latter requirement is equivalent to membership of φ in analytic BMO; **but** $H^\infty \subseteq BMO$**!** Hence, the last line in the chain of inequalities is bounded by a multiple of $\|f\|_{H^2}^2$.

From the first expression of the H^2 norm and after computing a geometric series, one finds that the formula for the reproducing kernel of H^2. If $z \in \Delta$, then

$$f(z) = < f, k_z >_{H^2}$$

with

$$k_z(w) = k(z,w) = \frac{1}{1-\overline{z}w}, \ \|k_z\|_{H^2}^2 = k(z,z) = \frac{1}{1-|z|^2}. \tag{0.3}$$

(ii) The Drury-Arveson space. The *Drury-Arveson space* $DA = DA_d$ is the reproducing Hilbert space obtained by replacing the product of complex numbers in (0.3) by a scalar product of complex vectors in $\mathbb{C}^d$ (hence, $DA_1 = H^2$):

$$k_z(w) = k(z,w) = \frac{1}{1-\overline{z}\cdot w}, \ z, w \in \mathbb{C}^d, \ |z|, |w| < 1. \tag{0.4}$$

The norm of a function $f(z) = \sum_{n\in\mathbb{N}^d} a_n z^n$ (multiindex notation is here used) can be expressed in various ways:

$$\begin{aligned}\|f\|_{DA}^2 &= \sum_n |a_n|^2 \frac{n!}{|n|!} \\ \text{(if } \sigma = 1/2) &\approx \text{(terms taking care of zeros of } f \text{ and its derivatives at the origin)} \\ &+ \int_{\mathbb{B}=\{|z|<1\}} \left|(1-|z|^2)^m (z\cdot\partial)^m f(z)\right|^2 (1-|z|^2)^{2\sigma} \frac{dm(z)}{(1-|z|^2)^{d+1}},\end{aligned} \tag{0.5}$$

where dm is Lebesgue measure in $\mathbb{B}$, $z \cdot \partial = \sum_{j=1}^{d} z_j \frac{\partial}{\partial z_j}$ and m is any sufficiently large integer. The equivalence of the norms holds with constants that depend on the dimension d. If we are interested in the *infinite dimansional* Drury-Arveson space DA_∞, we can not use the Sobolev-like norm as it is.

The space DA was introduced by Drury [11] in 1978 in connection with a version of the von Neumann inequality for rows of commutinng vectors. Let $A = (A_1, \dots, A_d) : H \to H^d$ be a vector of operators such that (i) $[A_j, A_k] = 0$: any two of them commute; and, (ii) $\sum_{j=1}^{d} |A_j h|^2 \le |h|^2$: A is a contraction. Let $p(z) = p(z_1, \dots, p_d)$ be a polynomial. Drury proved that

$$|||p(A)|||_{\mathcal{B}(H)} \le |||M_p|||_{\mathcal{B}(DA)} =: ||p||_{\mathcal{M}(DA)},$$

where $M_p : f \mapsto pf$ is the multiplication operator with multiplier p and $\mathcal{M}(DA)$ is the multiplier space of DA. Drury's inequality was redescovered and immersed in a larger context by Arveson [6] in 1998. Part of the theorem is the fact that equality holds for $A = M_z := (M_{z_1, \dots, M_{z_d}})$. Unfortunately, M_z is not a contraction, while it is M_z^*, the *backward shift.*

Drury's result raises the problem of characterizing the multipliers for DA. The second version for the norm of DA in (0.5), and a reasoning similar to the *wrong* one we used for the H^2 multipliers, show that a function φ analytic in the unit ball of complex space has multiplier norm which is approximately given by

$$\|\varphi\|^2_{\mathcal{M}(DA)} \approx \|\varphi\|^2_{H^\infty} + [d\mu_\varphi]_{CM(DA)}, \tag{0.6}$$

where

$$d\mu_\varphi(z) = \left|(1 - |z|^2)^m (z \cdot \partial)^m f(z)\right|^2 (1 - |z|^2)^{2\sigma} \frac{dm(z)}{(1 - |z|^2)^{d+1}} \ (\sigma = 1/2).$$

The *Carleson measure norm* $[d\mu]_{CM(DA)}$ of a positive measure μ on $\overline{\mathbb{B}}$ is the square of the norm of the imbedding $DA \hookrightarrow L^2(\mu)$. We are thus led to the problem of characterizing the Carleson measures for DA.

Let me mention a couple other reasons to be interested in Carleson measures.

- **Boundary values of DA functions.** Suppose $\mathrm{supp}(\mu) \subseteq E$, E closed in $\partial\mathbb{B}$. If μ is a Carleson measure for DA, we can say that the boundary values of f exist in a quantitative way on E. Characterizing Carleson measures is then a first, important step towards a geometric understanding of the "exceptional sets" for DA.
- **Interpolating sequences.** Let $Z = \{z_n\}_{n \ge 0}$ be a sequence in $\mathbb{B}$. By the reproducing property of the kernel, the restriction map

$$R_Z : f \mapsto \left\{ \frac{f(z_n)}{\|k_{z_n}\|_{DA}} \right\}_{n \ge 0} \tag{0.7}$$

maps DA into ℓ^∞, boundedly. The sequence Z is *universally interpolating* if R_Z is both bounded and surjective from DA to ℓ^2. By the general theory of the Nevanlinna-Pick kernels, on which we are going to say something more below, the universally interpolating sequences for DA are the interpolating sequences for the multiplier space $\mathcal{M}(DA)$, where the latter means that the map

$$Q : \varphi \mapsto \{\varphi(z_n)\}_{n \ge 0}$$

maps $\mathcal{M}(DA)$ onto ℓ^∞.

Problem. Find a geometric characterization of the interpolating sequences for DA.

For the one-dimensional Hardy space, all proofs of the characterization of the interpolating sequences make use of Blaschke products, which are not available in higher dimensions. On the other hand, the natural separation (necessary) condition for interpolating sequences is too weak to make room for the constructions which work for the unweighted Dirichlet space. See the monograph [17] for a beautiful overview of the topic.

(iii) Carleson measures for DA and other function spaces. As a first step towards the characterization of the Carleson measure for DA, we prove the following general lemma, which reduces the imbedding problem for rigid holomorphic functions to an integral inequality for flexible L^2 ones.

Lemma 1. *Let H be a Hilbert space of measurable functions on a metric space X having reproducing kernel $\{j_x\}_{x\in X}$ and containing the constant functions. Let Id be the identity operator.*

A measure $\mu \geq 0$ on X satisfies the Carleson imbedding

$$Id : H \hookrightarrow L^2(\mu)$$

if and only if the inequality

$$\int_X \left(\int_X \Re j(x,y) g(y) d\mu(y)\right) g(x) d\mu(x) \leq C \int_X g(t)^2 d\mu(t)$$

holds for all $g \geq 0$ and some $C > 0$. The best constant C in the inequality is comparable (with universal constants) with $|||Id|||_{\mathcal{B}(H,L^2(\mu))}$.

The proof is easy after we observe that the adjoint operator $Id^* = \Theta : L^2(\mu) \to H$ of Id has the form

$$\begin{aligned} \Theta g(x) &= <\Theta g, j_x>_H = <g, j_x>_{L^2(\mu)} \\ &= \int_X \overline{j(x,y)} g(y) d\mu(y). \end{aligned}$$

The norm inequality for Θ is then

$$\begin{aligned} C \cdot \int_X |g|^2 d\mu &\geq \|\Theta g\|_H^2 = <\Theta g, \Theta g>_H \\ &= <\int_X j_y g(y) d\mu(y), \int_X j_x g(y) d\mu(x)>_H \\ &= \int_X \left(\int_X <j_y, j_x>_H g(y) d\mu(y)\right) \overline{g(x)} d\mu(x) \\ &= \int_X \left(\int_X j(y,x) g(y) d\mu(y)\right) \overline{g(x)} d\mu(x). \end{aligned}$$

Testing the resulting integral inequality first on real, then just positive functions g, shows that the real part of the kernel only is essentially involved, and that testing on positive g's suffice.

If we want to apply Carleson measures to boundary values, we have to work with measures which are supported on the completion of X, which contains points at which evaluation might not be bounded. This is the case of most Hilbert function spaces, inculding DA. We are not going to discuss here the relatively modest technical problems involved in this extension.

In the Drury-Arveson case we have then to study the weighted inequality of Lemma 1 for the kernel

$$h(z,w)=\Re\left(\frac{1}{1-\overline{z}\cdot w}\right).$$

The good feature of the Lemma is that $h\geq 0$ on $\mathbb{B}$. The trouble is that $\frac{1}{h(z,w)}$ does not define a quasi-distance on $\partial\mathbb{B}$.

In order to better contextualize the problem, we consider DA as the $\sigma=1/2$ case in the scale of the Hilbert function spaces having reproducing kernel

$$k_\sigma(z,w)=\begin{cases}\frac{1}{(1-\overline{z}\cdot w)^{2\sigma}}, & \text{if } \sigma>0,\\ \log\left(\frac{1}{1-\overline{z}\cdot w}\right), & \text{if } \sigma=0.\end{cases}$$

The norms in these spaces are comparable with the last expression appearing in (0.5).

The kernel k_σ has nonnegative real part if $0\leq\sigma\leq 1/2$, while $\Re k_\sigma\approx|k_\sigma|$ if $0\leq\sigma<1/2$. In the family H_σ of Hilbert spaces so obtained, $H_{1/2}=DA$ is what we are intersted in; H_0 is the holomorphic Dirichlet space, which is the only holomorphic Hilbert space which is invariant under composition with conformal automorphisms of the complex ball; $H_{d/2}$ is the Hardy space; and H_σ is a weighted Bergman space for $\sigma>d/2$. The reason why we care about the metric properties of the kernel is that they bring in powerful tools from potential theory.

A theorem characterizing Carleson measures for DA, which puts together results in [3] and [18], is presented below. Define a metric on $\partial\mathbb{B}$ by setting $\rho(z,w)=|1-\overline{z}\cdot w|$. If $B=B(z,r)$ is the ball centered at $z\in\partial\mathbb{B}$ having radius $r>0$, let let $S(B)=\{w\in\overline{\mathbb{B}}:\ |1-\overline{z}\cdot w|\leq r\}$. The metric ρ and the "Carleson boxes" $S(B)$ can be interpreted in terms of the invariant Bergman metric on the unit ball.

Theorem 1 (Characterization of the Carleson measures for the DA space.)**.** *A measure μ on $\overline{\mathbb{B}}$ is a Carleson measure for DA if and only if it satifies the Carleson-type condition*

$$\mu(S(B))\leq C|B|$$

and the following condition holds for one (hence for all) $1\leq p<\infty$*:*

$$\int_{S(B)}\left(\int_{S(B)}\Re k(x,y)d\mu(y)\right)^p d\mu(x)\leq C_p\mu(S(B))$$

Part of the proofs consists in showing that the latter condition is equivalent to the estimate

$$\int_{\overline{\mathbb{B}}}\left|\int_{\overline{\mathbb{B}}}\Re k(x,y)g(y)d\mu(y)\right|^p d\mu(x)\leq C'_p\int_{\overline{\mathbb{B}}}|g|^p d\mu. \tag{0.8}$$

The proof in [3] makes use of the explicit expression of the kernel, while the one in [18] interprets the integral inequality (0.8) as an integral inequality for a "singular kernel" for the possibly non-doubling measure μ, and brings in the Menger curvature and other sophisticated machinery recently used in order to solve the Painlevé problem.

A still more general approach to the Carleson measures problem for the spaces H_σ, which works even in the missing range $1/2 < \sigma < d/2$, is in the recent [20].

(iv) The geometry of the DA kernel. For a detailed account of the geometry of the DA kernel, see [4]. Here we will content ourselves to give a sketch of the reasoning, implicitely assuming some knowledge of the geometry of the unit ball in $\mathbb{C}^d$. After Kelvin-transforming $\mathbb{B}$ onto the Siegel domain

$$\mathcal{U} = \{\zeta = (\zeta', \zeta_d) \in \mathbb{C}^{d-1} \times \mathbb{C} : \ \Im\zeta_d \geq |\zeta'|^2\},$$

the DA kernel becomes

$$k_{\mathcal{U}}(\zeta, \xi) = \frac{\overline{(i + \zeta_d)}(i + \xi_d)}{4 \cdot r(\zeta, \xi)}, \ r(\zeta, \xi) = \frac{i}{2}\overline{\zeta_d}\xi_d - \overline{\zeta'} \cdot \xi'. \tag{0.9}$$

We can define $h(\zeta, \xi) = 1/r(\zeta, \xi)$ to be the DA kernel for $\mathcal{U}$: the conjugating factors appearing in (0.9) have no deep impact on the analysis of the kernel. It is convenient to change to Heisenberg coordinates:

$$\zeta = (\zeta', \zeta_d) = [z, t; r] = [\zeta'; \Re\zeta_d \Im\zeta_d - |\zeta'|^2] \in \mathbb{C}^{d-1} \times \mathbb{R} \times (0, +\infty).$$

For $P = [z, t; r]$ and $Q = [w, s; q]$ the kernel h becomes

$$h(P, Q) = 2\frac{r + q + |z - w|^2 - i(t - s - 2\Im(\overline{z} \cdot w))}{(r + q + |z - w|^2)^2 + (t - s - 2\Im(\overline{z} \cdot w))^2}. \tag{0.10}$$

Introducing on the factor $\mathbb{C}^{d-1} \times \mathbb{R} = \mathbb{H} \ni (z, t)$ the Heisenberg group product,

$$(z, t) \cdot (w, s) = (z + w, t + s + 2\Im(z \cdot \overline{w})),$$

the kernel reveals its convolutional nature:

$$h([z, t; r], [w, s; q]) = 2\varphi_{r+p}\left((w, s)^{-1} \cdot (z, t)\right),$$

where

$$\varphi_r((z, t)) = \frac{r + |z|^2 - it}{(r + |z|^2)^2 + t^t}.$$

The real number $r > 0$ plays here the role of "scale", if we are interested in the Heisenberg group geometry; or, equivalently, keeps track of the dilation structure of $\mathbb{H}$. (The dilation with factor λ on $\mathbb{H}$ acts as the map $(z, t) \mapsto (\lambda z, \lambda^2 t)$).

We are now ready for a geometric intepretation of $\Re\varphi_0$, from which a geometric interpretation of h easily follows by convolution and Heisenberg scaling. The expression $\|(z, t)\| = (|z|^2 + t^2)^{1/4}$ defines a left invariant distance δ on $\mathbb{H}$ by $\delta((z, t), (w, s)) = \|(w, s)^{-1} \cdot (z, t)\|$. Now, the real part of φ_0 has a denominator which is $\delta((z, t), (0, 0))^4$, which defines a quasi-distance. The numerator $|z|^2$ can be interpreted as

$$|z|^2 = \delta((z, t), t - \text{axis})^2 := \inf\{\delta((z, t), (0, s))^2 : \ s \in \mathbb{R}\}.$$

In other terms, $|z - w|$ measures the Heisenberg distance between the Hopf fibers passing through $(z, 0)$ and $(w, 0)$. In [4] the distance δ is compared with the Bergman distance in the Siegel domain: the relation is analogous to that between the hyperbolic distance in the upper-half plane and the Euclidean geometry of its boundary.

The punchline of the considerations we have made is that $1/\varphi_0((w,s)^{-1}\cdot(z,t))$ *does not* define a quasi-distance between the points (z,t) and (w,s) in $\mathbb{H}$; hence some essential tools of potential theory can not be easily transplanted to study the space DA.

However, the kernel exhibits a rich geometry, and it would be interesting to see which kind of potential theory one can develop for it. In an admittedly vague fashion, we can state this as a question.

Problem. Develop a potential theory for the Drury-Arveson kernel and use it to study the exceptional sets for functions in DA.

A closed set $E\subseteq\partial\mathbb{B}$ is *exceptional* for DA if there is a function f in DA such that

$$\lim_{r\to 1^-} f(r\zeta)$$

does not exist for all ζ in E. By a potential theory related with the DA kernel we mean, roughly speaking, a potential theory in which the exceptional sets are the closed sets of zero capacity. See Chapter 2 of [1] for an introduction to very general versions of potential theory.

(v) Nehary-type theorems? In the context of the Hardy space, even if we are just interested in the Hilbert space H^2, we are forced to consider a number of other related Banach function spaces. We have seen that $H^\infty=\mathcal{M}(H^2)$ is the multiplier space of H^2. Another space naturally arising in the study of H^2 is H^1:

$$\begin{aligned}\|f\|_{H^1} &= \sup_{0\le r<1}\frac{1}{2\pi}\int_{-\pi}^{+\pi}|f(re^{it})|dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi}|f(e^{it})|dt,\end{aligned}$$

with the usual provision on boundary values. From Cauchy-Schwarz inequality, $\|f\cdot g\|_{H^1}\le\|f\|_{H^2}\|g\|_{H^2}$, a fact we might summarize as $H^2\cdot H^2\subseteq H^1$. After factoring out Blaschke products, it is in fact easy to verify that $H^2\cdot H^2=H^1$. Fefferman's duality result can be then stated as sayng that, under H^2-coupling, the dual space of $H^2\cdot H^2$ is the space $\chi(H^2)=BMO$ of those functions φ such that $d\mu_\varphi$ is a Carleson measure for H^2.

Similar results have been proved in other cotexts. See [5] for the case of the Dirichlet space on the unit disc.

In the Drury-Arveson space, due to the lack of Blaschke factors, we replace the product norm by a "weak product" norm:

$$\|f\|_{DA\odot DA}=\inf\{\sum_j\|a_j\|_{DA}\|b_j\|_{DA}:\ \sum_j a_jb_j=f\}.$$

See the important [9], where weak products were introduced, studied and applied. The analog of the BMO norm is

$$\|\varphi\|_{\chi(DA)}=[d\mu_\varphi]_{CM(DA)}^{1/2}.$$

If $d=1$, $DA\odot DA=H^2\odot H^2=H^2\cdot H^2=H^1$. It is rather easy to see that, under DA coupling, $\chi(DA)\subseteq[DA\odot DA]^*$.

Problem. Is it true that $\chi(DA) \subseteq [DA \odot DA]^*$?

(vi) The Drury-Arveson space and the complete Nevanlinna-Pick property. The Nevanlinna-Pick property shares with the von Nemann inequality the leading role among the good reasons to be interested in the Drury-Arveson space. Here, I follow closely the very nice exposition on [2].

Consider the following interpolation problem for a Hilbert space H of functions on a space X, having reproducing kernel k.

(IP) Given values $w_1, \dots, w_n$ in the complex unit disc Δ and $x_1, \dots, x_n$ in X; is there a function φ in the multiplier algebra $\mathcal{M}(H)$ of H such that $\|\varphi\|_{\mathcal{M}(H)} \le 1$ and $\varphi(x_j) = w_j$ for $1 \le j \le n$?

It is easy to find a *necessary* condition for this in terms of positivity of a certain matrix. First, we show that the conjugate values of a multiplier are eigenvalues of the multiplication operator's adjoint.

Proposition 1. *If φ is a multiplicator of H and M_φ^* denotes the adjoint of the corresponding multiplication operator, then*

$$M_\varphi^* k_x = \overline{\varphi(x)} k_x.$$

(By the way, an immediate consequence of the proposition is that $\|\varphi\|_{\mathcal{M}(H)} \ge \sup\{|\varphi(x)| : \ x \in X\}$, completing the discussion on Hardy multipliers in (i).) The proof is elementary:

$$\begin{aligned} M_\varphi^* k_x(y) &= < M_\varphi^* k_x, k_y >_H \\ &= \overline{< k_x, M_\varphi k_y >_H} \\ &= \overline{< M_\varphi k_y, k_x >_H} = \overline{< \varphi k_y, k_x >_H} = \overline{(\varphi k_y)(x)} = \overline{\varphi(x) k_y(x)} \\ &= \overline{\varphi(x)} k_x(y). \end{aligned}$$

Proposition 2. *If the interpolating problem (IP) has a solution, then the matrix $R := [k(x_j, x_k)(1 - \overline{w_j} w_k)]_{j,k=1}^n$ is positive semidefinite.*

In fact, a simple calculation implementing the inequality $\|\varphi\|_{\mathcal{M}(H)} \le 1$ gives:

$$\begin{aligned} \sum_{j,j} k(x_j, x_k) c_j \overline{c_k} &= \sum_{j,j} < k_{x_j}, k_{x_k} >_H \ c_j \overline{c_k} \\ &= \| \sum_j k_{x_j} c_j \|_H^2 \\ &\ge \| M_\varphi^* \left(\sum_j k_{x_j} c_j \right) \|_H^2 \\ &= \| \left(\sum_j \overline{\varphi}(x_j) k_{x_j} c_j \right) \|_H^2 \\ &\quad \text{by Proposition 1} \\ &= \sum_{j,j} k(x_j, x_k) \overline{\varphi(x_j)} \varphi(x_k) c_j \overline{c_k} \end{aligned}$$

$$= \sum_{j,j} k(x_j, x_k)\overline{w_j} w_k c_j \overline{c_k}.$$

The Hilbert space H has the Nevanlinna-Pick property if the condition in Proposition 2 is also *sufficient* to solve the interpolating problem (IP). It has the *complete Nevanlinna-Pick property* if a vector analog of (IP) can be solved.

Historically, the Nevanlinna-Pick property was first considered and proved for the Hardy space H^2_H, independently by Nevanlinna and Pick. The weighted Dirichlet spaces have the complete NP property in all complex dimensions, and this includes the DA space. The NP property is not stable under change of Hilbertian norm, so one should really specify the inner product one is working with. The Bergman spaces do not have the Nevanlinna-Pick property.

A result of Agler and McCarthy interlaces the complete NP property and DA.

Theorem 2. *[Agler-McCarthy 2000] Suppose H is a function Hilbert space on X with irreducible kernel k: $k(x,y) \neq 0$ for all x, y in X. Then, k has the complete Nevanlinna-Pick property if and only if there is a (possibly infinite) cardinal d, an injective function $f : X \to \mathbb{B}_m$ (the open unit ball in $\mathbb{C}^d$) and a nowhere vanishing function $\delta : X \to \mathbb{C}$ such that*

$$k(x,y) = \frac{\overline{\delta(x)}\delta(y)}{1 - \overline{f(x)}f(y)}.$$

Moreover, the map $k_x \mapsto \overline{\delta(x)} k^{DA_d}_{f(x)}$ extends to an isometry of H into δDA_d.

I will be vague on the exact nature of the infinite dimensional objects appearing in the statement. What is important is that many function HIlbert spaces with the complete NP property imbed, in fact, in infinite dimansional versions of the DA space. It would then be highly desirable to have a theory of DA in which the results are optimal with respect to dimension.

Problem. The constants appearing in Theorem 1 depend on the dimension d of the space. The Drury-Arveson space has an important meaning in its infinite dimensional version, however, in the context of the reproducing kernel Hilbert spaces satisfying the complete Nevanlinna-Pick property (see below). It would much desirable, then, having a dimensionless characterization of the Carleson measures for DA.

(vii) The Corona Theorem. We close these notes with a recent major achievement in the theory of DA: the Corona Theorem for $\mathcal{M}(DA_d)$, with finite d ([10]).

Theorem 3 (Corona Theorem.)**.** *Suppose $g_1, \dots, g_N$ are functions in $\mathcal{M}(DA)$ such that*

$$\sum_{j=1}^{N} |g_j(z)| \geq c > 0 \text{ on } \mathbb{B}.$$

Then, there are functions $f_1, \dots, f_N$ in $\mathcal{M}(DA)$ such that

$$\sum_{j^1}^{N} f_j g_j = 1.$$

In the one-dimensional, Hardy space case, the theorem was a striking result by L. Carleson [8]. In functional analytic terms, the theorem can be interpreted as saying the all the maximal ideals of the multiplicative algebra $\mathcal{M}(DA)$ lie in the closure (with respect to the Gelfand topology) of the unit ball $\mathbb{B}$, where each z_0 in $\mathbb{B}$ is identified with $m_{z_0} = \{\varphi \in \mathcal{M}(DA) : \varphi(z_0) = 0\}$. See e.g. [13] for a thorough discussion of Carleson's Corona Theorem and its proofs.

Some references. A very nice account of DA theory from the viewpoint of operator theory is the book by Agler and McCarthy [2]. The article [11] by Drury, where the space DA first appeared, is very readable and the proof of the multivariable von Neumann inequality is less that two pages long. The very well-written and comprehensive article [6] by Arveson prompted the current interest in the theory of the DA space. Some more problems on DA, including a "flat“ and a discrete variation thereof, is in [4].

Finally, some apparently off-topic titles. The preprint [14] by Marshall and Sundberg, where interpolating sequences for the Dirichlet space are characterized, in noteworthy for its clarity, mantaining an elegant equilibrium between the operator theoretic arguments and the hard analysis which is needed at some crucial point of the proof. It is probably the best place to start from in a quest for the interpolating sequences for DA. The article [15] by Nagel, Rudin and Shapiro, plays a similar role with regards to exceptional sets and convergence to the boundary. Beurling's landmark paper [7] is where all stories concerning holomorphic function spaces and operator theory begin.

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Dipartimento di Matematica, Universita di Bologna, 40127 Bologna, ITALY
E-mail address: nicola.arcozzi@unibo.it