On finite groups acting on spheres and finite subgroups of orthogonal groups

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Abstract. *This is a survey on old and new results, and also an introduction to some related basic notions and concepts, on which my talk at the workshop "Varietà reali e complesse: Geometria, Topologia e Analisi armonica", Pisa, Scuola Normale Superiore, 28.2. - 3.3. 2013, was based. I discuss finite groups acting on low-dimensional spheres and homology spheres, comparing with the finite subgroups of the corresponding orthogonal groups, and also finite simple groups acting on spheres and homology spheres of arbitrary dimension. A version of this has appeared in [Z3] (see also [Z2] for a discussion of the cases of integer and mod 2 homology spheres).*

1. Introduction

We are interested in the class of finite groups which admit an orientation-preserving action on a sphere S^n of a given dimension n. All actions in the present paper will be faithful and orientation-preserving (but no necessarily free). So formally an action of a finite group G is an injective homomorphism from G into the group of orientation-preserving homeomorphism of S^n; informally, we will consider G as a group of orientation-preserving homeomorphisms of S^n and distinguish various types of actions:

topological actions: G acts by homeomorphisms;

smooth actions: G acts by diffeomorphisms;

linear (orthogonal) actions: G acts by orthogonal maps of $S^n \subset \mathbb{R}^{n+1}$ (that is, G is a subgroup of the orthogonal group $\mathrm{SO}(n+1)$); or, more generally, any topological action which is conjugate to a linear action;

locally linear actions: topological actions which are linear in regular neighbourhoods of fixed points of any nontrivial subgroup.

It is well-known that smooth actions are locally linear (by the existence of equivariant regular neighbourhoods, see [Bre]; see also the discussion in the next two sections for some examples).

The reference model for finite group actions on S^n is the orientation-preserving orthogonal group $\mathrm{SO}(n+1)$; in fact, the only examples of actions which can easily be seen are linear actions by finite subgroups of $\mathrm{SO}(n+1)$. A rough general guiding line is then the following:

Motivating naive conjecture: Every action of a finite group on S^n is linear (that is, conjugate to a linear action). More generally, how far can an action by homeomorphisms or diffeomorphisms be from a linear action?

It is one of the main features of linear actions that fixed point sets of single elements are standard unknotted spheres S^k in S^n (that is, intersections of linear subspaces $\mathbb{R}^{k+1}$ of $\mathbb{R}^{n+1}$ with S^n). We note that it is a classical and central result of Smith fixed point theory that, for an action of a finite p-group (a group whose order is a power of a prime p) on a mod p homology n-sphere (a closed n-manifold with the mod p homology of S^n), fixed point sets are again mod p homology spheres (see [Bre]; one has to consider homology with coefficients in the integers mod p here since this does not remain true in the setting of integer homology spheres).

We note that for *free actions* on S^n (nontrivial elements have empty fixed point sets), the class of finite groups occurring is very restricted (they have periodic cohomology, of period $n+1$, see [Bro]). On the other hand, for not necessarily free actions, all finite groups occur for some n (by just considering a faithful, real, linear representation of the finite group). Two of the motivating problems of the present survey are then the following:

Problems. i) Given a dimension n, determine the finite groups G which admit an action on a sphere or a homology sphere of dimension n (comparing with the class of finite subgroups of the orthogonal group $\mathrm{SO}(n+1)$).

ii) Given a finite group G, determine the minimal dimension of a sphere or homology sphere on which G admits an action (show that it coincides with the minimal dimension of a linear action of G on a sphere).

We close the introduction with a general result by Dotzel and Hamrick ([DH]), again for finite p-groups: If G is a finite p-group acting smoothly on a mod p homology n-sphere then G admits also a linear action on S^n such that the two actions have the same dimension function for the fixed point sets of all subgroups of G.

In the next sections we will discuss finite groups acting on low-dimensional spheres, starting with the 2-sphere S^2.

2. Finite groups acting on S^2 and finite subgroups of SO(3).

Let G be a finite group of (orientation-preserving) homeomorphisms of the 2-sphere S^2. It is a classical result of Brouwer and Kerekjarto from 1919 that such a finite group action on a 2-manifold is *locally linear*: each fixed point of a nontrivial element has a regular neighbourhood which is a 2-disk on which the cyclic subgroup fixing the point acts as a standard orthogonal rotation on the 2-disk. This implies easily that the quotient space S^2/G (the space of orbits) is a again a 2-manifold, or better a *2-orbifold* of some signature $(g; n_1, \ldots, n_r)$: an orientable surface of some genus g, with r branch points of orders $n_1, \ldots, n_r$ which are the projections of the fixed points of the nontrivial cyclic subgroup of G (of orders $n_i > 1$).

The projection $S^2 \to S^2/G$ is a *branched covering*. We choose some triangulation of the quotient orbifold S/G such that the branch points are vertices of the triangulation, and lift this triangulation to a triangulation of S^2; then the projection $S^2 \to S^2/G$ becomes a simplicial map. If the projection p is a covering in the usual sense (unbranched, i.e. without branch points), then clearly the Euler characteristic χ behaves multiplicatively, i.e. $2 = \chi(S^2) = |G|\ \chi(S^2/G) = |G|\ (2-2g)$ (just multiplying Euler characteristics with the order $|G|$ of G). In the case with branch points, we can correct this by subtracting $|G|$ for each branch point and then adding $|G|/n_i$ (the actual number of points of S^2 projecting to the i'th branch point), obtaining in this way the classical *formula of Riemann-Hurwitz*:

$$2 = \chi(S^2) = |G|\ (2 - 2g - \sum_{i=1}^{r}(1 - \frac{1}{n_i})).$$

It is easy to see that the only solutions with positive integers of this equation are the following (first two columns):

$(0; n, n)$,	$\lvert G\rvert = n$;	$G \cong \mathbb{Z}_n$	cyclic;
$(0; 2, 2, n)$,	$\lvert G\rvert = 2n$;	$G \cong \mathbb{D}_{2n}$	dihedral;
$(0; 2, 3, 3)$,	$\lvert G\rvert = 12$;	$G \cong \mathbb{A}_4$	tetrahedral;
$(0; 2, 3, 4)$,	$\lvert G\rvert = 24$;	$G \cong \mathbb{S}_4$	octahedral;
$(0; 2, 3, 5)$,	$\lvert G\rvert = 60$;	$G \cong \mathbb{A}_5$	dodecahedral.

We still have to identify the groups G. For this, we first determine the finite subgroups of the orthogonal group SO(3) and suppose that the action of G on S^2 is orthogonal. Of course the possibilities for the signatures of the quotient orbifolds S^2/G and the orders $|G|$ remain the same, and for orthogonal actions one can identify the groups now as the orientation-preserving symmetry groups of the platonic solids. As an example, if the signature is (0; 2,3,5) and $|G| = 60$, one considers a fixed point P of a cyclic subgroup $\mathbb{Z}_5$ of G and the five fixed points of subgroups $\mathbb{Z}_3$ closest to P on S^2; these are the vertices of a regular pentagon on S^2 which is one of the twelve pentagons of a regular dodecahedron projected to S^2, invariant under the action of G. Hence G, of order 60, coincides with the orientation-preserving isometry group of the regular dodecahedron, the dodecahedral group $\mathbb{A}_5$; see [W, section 2.6] for more details. In this way one shows that the finite subgroups of SO(3) are, up to conjugation, exactly the *polyhedral groups* as indicated in the list above: cyclic $\mathbb{Z}_n$, dihedral $\mathbb{D}_{2n}$, tetrahedral $\mathbb{A}_4$, octahedral $\mathbb{S}_4$ and dodecahedral $\mathbb{A}_5$.

As a consequence, returning to topological actions of finite groups G on S^2, the topological orbifolds S^2/G in the above list are geometric since they are homeomorphic exactly to the quotients of S^2 by the polyhedral groups. Hence the topological orbifolds S^2/G have a spherical orbifold structure (a Riemannian metric of constant curvature 1, with singular cone points of angles $2\pi/n_i$); lifting this spherical structure to S^2 realizes S^2 as a spherical manifold (i.e., with a Riemannian metric of constant curvature 1, without singular points). Since the spherical metric on S^2 is unique up to isometry this gives the standard Riemannian S^2, and G acts by isometries now. Hence every topological action of a finite group G on S^2 is conjugate to an orthogonal action, that is finite group actions on S^2 are linear (this can be considered as the *orbifold geometrization in dimension 2*, in the spherical case).

Concluding and summarizing, finite group actions on S^2 are *locally linear*, and then also *linear*; the finite groups occurring are exactly the polyhedral groups, and every topological action of such a group is geometric (or linear, or orthogonal), i.e. conjugate to a linear action.

3. Finite groups acting on S^3 and finite subgroups of SO(4).

3.1. Geometrization of finite group actions on S^3

We consider actions of a finite group G on the 3-sphere S^3 now. The first question is if such a topological action is *locally linear*; by the normal form for orthogonal matrices, in dimension 3 this means that an element with fixed points acts locally as a standard rotation around some axis (the orientation-preserving case). Unfortunately this is no longer true in dimension 3; after a first example of Bing from 1952 in the orientation-reversing case, Montgomery-Zippen 1954 gave examples also of orientation-preserving cyclic group actions on S^3 with "wildly embedded fixed point sets", i.e. with fixed points sets which are locally not homeomorphic to the standard embedding of S^1 in S^3. Obviously such actions are not locally linear, and in particular cannot be conjugate to smooth or linear actions.

We will avoid these wild phenomena in the following by concentrating on *smooth* or *locally linear* actions. Suppose now that a cyclic group $G \cong Z_p$, for a prime p, acts locally linear on S^3 with nonempty fixed point set; then it acts locally as a standard rotation around an axis, and by compactness this axis closes globally to an embedded knot $K \cong S^1$ in S^3. We note that also for a topological action of G, by general Smith fixed point theory the fixed point set of G is a knot K, i.e. an embedded S^1 in S^3. If the action is locally linear, K is a tame knot as considered in classical knot theory (smooth or polygonal embeddings), otherwise K is a wild knot leading to the different field of wild or Bing topology (see [Ro, chapter 3.I] for such wild phenomena in dimension 3).

So we will consider only *locally linear* actions in the following. Globally, the question arises then which knots K can occur as the fixed point set of an action of a cyclic group on S^3; it is easy to see that the action of $G \cong Z_p$ is *linear* (conjugate to a linear action) if and only if K is a trivial knot (unknotted, i.e. bounding a disk in S^3). The classical *Smith conjecture* states that K is always a trivial knot, and that consequently locally linear actions of cyclic groups are linear. A positive solution of the Smith conjecture was the first major success of Thurston's geometrization program for 3-manifolds (see [MB]). This has been widely generalized by Thurston then who showed that finite nonfree group actions on closed 3-manifolds are build from geometric actions (*orbifold geometrization in dimension 3*), and recently by Perelman also for free actions of finite groups (*manifold geometrization in dimension 3*). As a consequence, every finite group acting smoothly or locally linearly on S^3 is geometric, i.e. conjugate to an orthogonal or linear action.

Concluding, finite group actions on S^3 are not locally linear, in general, but smooth or locally linear actions are linear; in particular, the finite groups acting smoothly or locally linearly on S^3 are exactly the finite subgroups of the orthogonal group SO(4).

In the remaining part of this section we discuss the finite subgroups of the orthogonal group SO(4), starting with the relation between SO(3) and the unit quaternions.

3.2. The orthogonal group SO(3) and the unit quaternions S^3.

The orthogonal group SO(3) is homeomorphic to the real projective space $\mathbb{RP}^3$ of dimension 3. In fact, by the normal form for orthogonal 3×3 matrices, such a matrix induces a clockwise rotation of the unit 3-ball in $\mathbb{R}^3$ around some oriented axis or diameter; parametrising the diameter by an rotation angle from $-\pi$ to π, one obtains SO(3) by identifying diametral points on the boundary S^2 of the 3-ball (since $-\pi$ and π give the same rotation), and consequently SO(3) is homeomorphic to $\mathbb{RP}^3$.

Hence the universal covering of $\mathrm{SO}(3) \cong \mathbb{RP}^3 \cong S^3/\langle \pm \mathrm{id} \rangle$ is the 3-sphere S^3. Considering S^3 as the unit quaternions, an orthogonal action of S^3 on the 2-sphere S^2 is obtained as follows. The unit quaternions S^3 act on itself by conjugation $x \to q^{-1}xq$, for a fixed $q \in S^3$; this action is clearly linear and also orthogonal. Since q fixes both poles 1 and -1 in S^3, it restricts to an orthogonal action on the corresponding equatorial 2-sphere S^2 in S^3, so this defines an element of the orthogonal group SO(3) and a group homomorphism $S^3 \to \mathrm{SO}(3)$ of Lie groups of the same dimension, with kernel ± 1; by standard facts about Lie groups, this is the universal covering of $\mathrm{SO}(3) \cong S^3/\langle \pm 1 \rangle$.

The finite subgroups of SO(3) are exactly the *polyhedral groups* $\mathbb{Z}_n, \mathbb{D}_{2n}, \mathbb{A}_4, \mathbb{S}_4$ and $\mathbb{A}_5$. Their preimages in the unit quaternions S^3 are the *binary polyhedral groups*

$$\mathbb{Z}_{2n},\ \mathbb{D}^*_{2n},\ \mathbb{A}^*_4,\ \mathbb{S}^*_4,\ \mathbb{A}^*_5$$

(cyclic, binary dihedral, binary tetrahedral, binary octahedral or binary dodecahedral). Since S^3 has a unique nontrivial involution -1, together with the cyclic groups of odd order these are exactly the finite subgroups of the unit quaternions S^3, up to conjugation. By right multiplication, they act freely and orthogonally on the 3-sphere S^3, and the quotient spaces are examples of spherical 3-manifolds; for example, $S^3/\mathbb{A}^*_5$ is the *Poincaré homology 3-sphere*.

We note that the Lie group S^3 has various descriptions: it occurs as the unitary group SU(2) over the complex numbers, as the universal covering group Spin(3) of SO(3) over the reals, and finally as the symplectic group Sp(1) over the quaternions (in fact, the unit quaternions).

3.3. The orthogonal group SO(4) as a central product $S^3 \times_{\mathbb{Z}_2} S^3$.

Passing to the orthogonal group SO(4) now, acting on the unit 3-sphere $S^3 \subset \mathbb{R}^4$, there is an orthogonal action of $S^3 \times S^3$ on S^3 given by $x \to q_1^{-1}xq_2$, for a fixed pair of unit quaternions $(q_1, q_2) \in S^3 \times S^3$. This defines again a homomorphism of Lie groups $S^3 \times S^3 \to \mathrm{SO}(4)$ of the same dimension, with kernel $\mathbb{Z}_2$ generated by $(-1, -1)$, so this is in fact the universal covering of the Lie group SO(4). In particular, the universal covering group Spin(4) of SO(4) is isomorphic to $S^3 \times S^3$, and SO(4) is isomorphic to the central product $S^3 \times_{\mathbb{Z}_2} S^3$ of two copies of the unit quaternions (the direct product with identified centers, noting that the center of the unit quaternions is isomorphic to $\mathbb{Z}_2$ generated by -1).

Identifying SO(4) with $S^3 \times_{\mathbb{Z}_2} S^3$, the finite subgroups of SO(4) are, up to conjugation, exactly the finite subgroups of the central products

$$P_1^* \times_{\mathbb{Z}_2} P_2^* \subset S^3 \times_{\mathbb{Z}_2} S^3$$

of two binary polyhedral groups P_1^* and P_2^*. The most interesting example of such a group is the central product $\mathbb{A}^*_5 \times_{\mathbb{Z}_2} \mathbb{A}^*_5$ of two binary dodecahedral groups which occurs as the orientation-preserving symmetry group of the regular 4-dimensional 120-cell (a fundamental domain for the universal covering group $\mathbb{A}^*_5$ of the Poincaré homology sphere $S^3/\mathbb{A}^*_5$ is a regular spherical dodecahedron, and 120 copies of this dodecahedron give a regular spherical tesselation of the 3-sphere S^3; the vertices of this tesselation are the vertices of the regular euclidean 120-cell in $\mathbb{R}^4$ whose faces are 120 regular dodecahedra.)

Concluding, the finite subgroups of SO(4) are exactly the subgroups of the central products $P_1^* \times P_2^*$ of two binary polyhedral groups P_1^* and P_2^*. It is then an algebraic exercise to classify the possible groups, up to isomorphism and up to conjugation (see [DV] for a list of the finite subgroups of SO(4), and also of O(4)).

4. Finite groups acting on S^4 and finite subgroups of SO(5).

As we have seen in the previous sections, finite group actions on the 2-sphere are locally linear, and then also linear. In dimension 3, finite group actions are not locally linear, in general, but by deep results of Thurston and Perelman, smooth or locally linear actions on the 3-sphere are linear. In dimension 4, also smooth or locally linear actions are no longer linear, in general. In fact it has been shown by Giffen in 1966 that the Smith conjecture fails in dimension 4, by constructing examples of smooth actions of a finite cyclic group on S^4 whose fixed point sets are knotted 2-spheres in S^4 (see [R, chapter 11.C]); in particular, such an action cannot be linear.

Restricting again to smooth or locally linear actions, we consider now the problem stated in the introduction: which finite groups G admit a smooth orientation-preserving action on the 4-sphere S^4; also, what are the finite subgroups of SO(5)?

Suppose that G is a finite group with an orientation-preserving, faithful, linear action on $S^4 \subset \mathbb{R}^5$; using the language of group representations, this means that G has a faithful, orientation-preserving, real representation in dimension 5. If such a representation is *reducible* (a direct sum of lower-dimensional representations), G is an orientation-preserving subgroup of a product of orthogonal groups $\mathrm{O}(3) \times \mathrm{O}(2)$ or $\mathrm{O}(4) \times \mathrm{O}(1)$, so one can reduce to lower dimensions.

Suppose that the representation is *irreducible* but *imprimitive*; this means that there is a decomposition of $\mathbb{R}^5$ into proper linear subspaces which are permuted transitively by the group. Since the dimension 5 is prime, these linear subspaces have to be 1-dimensional (such a representation is then called monomial). The group of orthogonal maps permuting the five factors $\mathbb{R}$ of $\mathbb{R}^5$ is the Weyl-group $W_5 = (\mathbb{Z}_2)^5 \rtimes \mathbb{S}_5$ of inversions and permutations of coordinates, i.e. the semidirect product of the normal subgroup $(\mathbb{Z}_2)^5$ generated by the inversions and the symmetric group $\mathbb{S}_5$ of permutations of the factors. Hence G is a subgroup of the Weyl-group $W = (\mathbb{Z}_2)^5 \rtimes \mathbb{S}_5$ in this case.

There remains the case of an *irreducible, primitive* representation; this is the main case which has been considered by various authors and for arbitrary dimension; a major problem here is to find the simple groups which admit such a representation (i.e., groups without a nontrivial proper normal subgroup). This leads into classical representation theory of finite groups, and we will not go further into it. In fact, we gave the above description mainly as a motivation for the next result on smooth or locally linear actions of finite groups on the 4-sphere.

Theorem 1. ([MeZ1]) *A finite group G with a smooth or locally linear, orientation-preserving action on the 4-sphere, or on any homology 4-sphere, is isomorphic to one of the following groups:*

i) an orientation-preserving subgroup of $\mathrm{O}(3) \times \mathrm{O}(2)$ *or* $\mathrm{O}(4) \times \mathrm{O}(1)$*;*

ii) an orientation-preserving subgroup of the Weyl group $W = (\mathbb{Z}_2)^5 \rtimes \mathbb{S}_5$*;*

iii) $\mathbb{A}_5$, $\mathbb{S}_5$, $\mathbb{A}_6$ *or* $\mathbb{S}_6$*;*

i') if G is solvable, a 2-fold extension of a subgroup of $\mathrm{SO}(4)$.

Note that the different cases of Theorem 1 are not mutually exclusive. The only indetermination remains case i'); here we conjecture that G is isomorphic to a subgroup of $\mathrm{O}(4)$ and hence to an orientation-preserving subgroup of the group $\mathrm{O}(4) \times \mathrm{O}(1)$ of case i); however the proof in this case is not completed at present since many different cases have to be considered, according to the list of the finite subgroups of $\mathrm{SO}(4)$ (see [DV]).

Corollary 1. *A finite group G which admits a smooth or locally linear, orientation-preserving action on a homology 4-sphere is isomorphic to a subgroup of* $\mathrm{SO}(5)$ *or, if G is solvable, to a 2-fold extension of a subgroup of* $\mathrm{SO}(4)$.

The symmetric group $\mathbb{S}_6$ acts orthogonally on $\mathbb{R}^6$ by permutation of coordinates, and also on its subspace $\mathbb{R}^5$ defined by setting the sum of the coordinates equal to zero (this is called the standard representation of $\mathbb{S}_6$), and hence on the unit sphere $S^4 \subset \mathbb{R}^5$. Composing the orientation-reversing elements by -id, one obtains an orientation-presering action of $\mathbb{S}_6$ on S^4 (alternatively, $\mathbb{S}_6$ acts on the 5-simplex by permuting its six vertices, and hence on its boundary which is the 4-sphere.)

For linear action, Theorem 1 and its proof easily give the following characterization of the finite subgroups of $\mathrm{SO}(5)$.

Corollary 2. *Let G be a finite subgroup of the orthogonal group* $\mathrm{SO}(5)$. *Then one of the following cases occurs:*

i) G is conjugate to an orientation-preserving subgroup of $\mathrm{O}(4) \times \mathrm{O}(1)$ *or* $\mathrm{O}(3) \times \mathrm{O}(2)$ *(the reducible case);*

ii) G is conjugate to a subgroup of the Weyl group $W = (\mathbb{Z}_2)^4 \rtimes \mathbb{S}_5$ *(the irreducible, imprimitive case);*

iii) G is isomorphic to $\mathbb{A}_5$, $\mathbb{S}_5$, $\mathbb{A}_6$ *or* $\mathbb{S}_6$ *(the irreducible, primitive case).*

See the character tables in [Con] or [FH] for the irreducible representations of the groups in iii) (e.g. $\mathbb{A}_5$ occurs as an irreducible subgroup of all three orthogonal groups $\mathrm{SO}(3)$, $\mathrm{SO}(4)$ and $\mathrm{SO}(5)$).

It should be noted that the proof of Corollary 2 is considerably easier than the proof of Theorem 1. For both Theorem 1 and Corollary 2 one has to determine the finite simple groups which act on a homology 4-sphere resp. which admit an orthogonal action on the

4-sphere. In the case of Theorem 1 this is based on [MeZ2, Theorem 1] which employs the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most 4 (see [Su2], [G1]). For the proof of Corollary 2 instead, this heavy machinery from the classification of the finite simple groups can be replaced by much shorter arguments from the representation theory of finite groups.

For solvable groups G instead, the proof of Theorem 1 is easier; here one can consider the Fitting subgroup of G, the maximal normal nilpotent subgroup, which is nontrivial for solvable groups. As a nilpotent group, the Fitting subgroup is the direct product of its Sylow p-subgroups, has nontrivial center and hence nontrivial cyclic normal subgroups of prime order. A starting point of the proof of Theorem 1 is then the following lemma which shows some of the basic ideas involved.

Lemma 1. *Let G be a finite group with a smooth or locally linear, orientation-preserving action on a homology 4-sphere. Suppose that G has a cyclic normal group $\mathbb{Z}_p$ of prime order p; by Smith fixed point theory, the fixed point set of $\mathbb{Z}_p$ is either a 0-sphere S^0 or a 2-sphere S^2 (i.e., a mod p homology sphere of even codimension).*

i) If the fixed point set of $\mathbb{Z}_p$ is a 0-sphere then G contains of index at most 2 a subgroup isomorphic to a subgroup of $\mathrm{SO}(4)$. *Moreover if G acts orthogonally on S^4 then G is conjugate to a subgroup of* $\mathrm{O}(4) \times \mathrm{O}(1)$.

ii) If the fixed point set of $\mathbb{Z}_p$ is a 2-sphere then G is isomorphic to a subgroup of $\mathrm{O}(3) \times \mathrm{O}(2)$. *Moreover if G acts orthogonally on S^4 then G is conjugate to a subgroup of* $\mathrm{O}(3) \times \mathrm{O}(2)$.

Proof. i) Since $\mathbb{Z}_p$ is normal in G, the group G leaves invariant the fixed point set S^0 of $\mathbb{Z}_p$ which consists of two points. A subgroup G_0 of index at most 2 of G fixes both points and acts orthogonally and orientation-preservingly on a 3-sphere, the boundary of a G_0-invariant regular neighborhood of one of the two fixed points.
If the action of G is an orthogonal action on the 4-sphere then G acts orthogonally on the equatorial 3-sphere of the 0-sphere S^0 and hence is a subgroup of $\mathrm{O}(4) \times \mathrm{O}(1)$, up to conjugation.

ii) The group G leaves invariant the fixed point set S^2 of $\mathbb{Z}_p$. A G-invariant regular neighbourhood of S^2 is diffeomorphic to the product of S^2 with a 2-disk, so G acts on its boundary $S^2 \times S^1$ (preserving its fibration by circles). Now, by the geometrization of finite group actions in dimension 3, it is well-known that every finite group action on $S^2 \times S^1$ preserves the product structure and is standard, i.e. is conjugate to a subgroup of its isometry group $\mathrm{O}(3) \times \mathrm{O}(2)$.
If G acts orthogonally on S^4 then the group G leaves invariant S^2, the corresponding 3-dimensional subspace in $\mathbb{R}^5$ as well as its orthogonal complement, so up to conjugation it is a subgroup of $\mathrm{O}(3) \times \mathrm{O}(2)$.

5. Higher dimensions

Relevant in the context of linear actions on spheres is the classical *Jordan number*: for each dimension n there is an integer $j(n)$ such that each finite subgroup of the complex linear group $\mathrm{GL}_n(\mathbb{C})$, and hence in particular also of its subgroup $\mathrm{SO}(n)$, has a normal abelian subgroup of index at most $j(n)$. The optimal bound for all n has recently been determined in [Col]: for $n \geq 71$ it is $(n+1)!$, realized by the symmetric group $\mathbb{S}_{n+1}$ which is a subgroup of $\mathrm{GL}_n(\mathbb{C})$; this requires the classification of the finite simple groups.

Whereas the Jordan bound is insignificant for abelian group, it implies that the order of a nonabelian simple groups acting linearly on S^n is bounded by $j(n+1)$; in particular, up to isomorphism there are only finitely many finite simple groups (always understood to be nonabelian in the following) which admit a faithful, linear action on S^n (or equivalently, have a faithful, real, linear representation in dimension $n+1$). For smooth or locally linear actions of finite simple groups on spheres and homology spheres, there is the following analogue.

Theorem 2. ([GZ]) *For each dimension n, up to isomorphism there are only finitely many finite simple groups which admit a smooth or locally linear action on the n-sphere, or on some homology sphere of dimension n.*

We note that any finite simple group admits many smooth actions on high-dimensional spheres which are not linear (conjugate to a linear action; see the survey [Da, section 7]).

It is then natural to ask whether the Jordan number can be generalized for all finite groups acting on homology n-spheres. Since, as noted in the introduction, finite p-groups admitting a smooth or locally linear action on some homology n-sphere admit also a linear action on S^n ([DH]), it is easy to generalize the Jordan number for nilpotent groups; so the Jordan number remains true for the two extreme opposite cases of nilpotent groups and simple groups, but at present we do not know such a bound for arbitrary finite groups.

The proof of Theorem 2 requires the classification of the finite simple groups; it permits to produce for each dimension n a finite list of finite simple groups which are the candidates for actions on homology n-spheres. For example, it is shown in [Z1], [MeZ2,3] that the only finite simple group which admits an action on a homology 3-sphere is the alternating group $\mathbb{A}_5$, and that the only finite simple groups acting on a homology 4-sphere are the alternating groups $\mathbb{A}_5$ and $\mathbb{A}_6$ (see also the survey [Z2]); already these low-dimensional results require heavy machinery from the classification of the finite simple groups. We shall consider the case of dimension 5 in Theorem 5.

Crucial for the proof of Theorem 2 is also a control over the minimal dimension of an action of a linear fractional group $\mathrm{PSL}_2(p)$ and a linear group $\mathrm{SL}_2(p)$ (the latter is the group of 2×2-matrices of determinant 1 over the finite field with p elements, the former its factor group by the central subgroup $\mathbb{Z}_2 = \langle \pm E_2 \rangle$); this is given by the following:

Theorem 3. ([GZ]) *For a prime $p \geq 5$, the minimal dimension of an action of a linear fractional group $\mathrm{PSL}_2(p)$ on a mod p homology sphere is $(p-1)/2$ if $p \equiv 1 \bmod 4$, and $p-2$ if $p \equiv 3 \bmod 4$, and these are also lower bounds for the dimension of such an action of a linear group $\mathrm{SL}_2(p)$.*

Whereas the groups $\mathrm{PSL}_2(p)$ admit linear actions on spheres of the corresponding dimensions (see e.g. [FH]), for the groups $\mathrm{SL}_2(p)$ the minimal dimension of a linear action on a sphere is $p-2$ resp. p which is strictly larger than the lower bounds given in Theorem 3, so the minimal dimension of an action on a homology sphere remains open here.

In analogy with Theorem 3, we need also the following result from Smith fixed point theory for elementary abelian p-groups ([Sm]).

Theorem 4. *The minimal dimension of a faithful, orientation-preserving action of an elementary abelian p-group $(\mathbb{Z}_p)^k$ on a mod p homology sphere is k if $p = 2$, and $2k-1$ if p is an odd prime.*

We present two proofs of Theorem 2. The first one is the short and conceptual proof from [GZ]. The second proof is more explicit, considering case by case the infinite series in the classification of the finite simple groups; it is longer and more technical but useful if one wants to determine the finite simple groups which are the candidates for an action in a fixed dimension n (see Theorem 5 for dimension 5).

Proof of Theorem 2. Fixing a dimension n, we have to exclude all but finitely many finite simple groups. By the classification of the finite simple groups, a finite simple group is one of 26 sporadic groups, or an alternating group, or a group of Lie type ([Con], [G1,2]). We can neglect the sporadic groups and have to exclude all but finitely many groups of the infinite series. Clearly, an alternating group $\mathbb{A}_m$ contains elementary abelian subgroups $(\mathbb{Z}_2)^k$ of rank k growing with the degree m, so Theorem 4 excludes all but finitely many alternating groups and we are left with the infinite series of groups of Lie type. Since every group of Lie type in characteristic $p \geq 5$ contains a subgroup isomorphic to either $\mathrm{SL}_2(p)$ or $\mathrm{PSL}_2(p)$, inside a root SL, the characteristic p is bounded by Theorem 3; also, root subgroups bound the field order by Theorem 4. This excludes all but finitely many groups from each series of Lie type and completes the proof of Theorem 2.

In the second proof of Theorem 2, we will consider the finite simple groups of Lie type case by case, excluding from each infinite series all by finitely many groups. We still need the following lemma whose proof is easy considering commutator subgroups. A finite *central extension* of G is a finite group with a central subgroup whose factor group is isomorphic to G; a group is *perfect* if its abelianization is trivial.

Lemma 2. *If a finite group G has a perfect subgroup H then any finite central extension of G contains a perfect central extension of H.*

We will apply Lemma 2 mainly when H is a linear group $\mathrm{SL}_2(q)$, for a prime power $q = p^k$ (that is, over the finite field with p^k elements); we note that, for $q \geq 5$ and different from 9, the only perfect central extension of the perfect group $\mathrm{SL}_2(q)$ is the group itself (see [H, chapter V.25]) (and the only nontrivial perfect central extension of $\mathrm{PSL}_2(q)$ is $\mathrm{SL}_2(q)$).

We consider first the *projective linear groups* $\mathrm{PSL}_m(q)$, for a prime power $q = p^k$. The group $\mathrm{PSL}_2(q)$ has subgroups $\mathrm{PSL}_2(p)$ and $(\mathbb{Z}_p)^k$ (the subgroup represented by all diagonal matrices with entries 1 on the diagonal, isomorphic to the additive group of the field with p^k elements), and by Theorems 3 and 4 only finitely many primes p and prime powers p^k can occur. If $m \geq 3$ instead, $\mathrm{PSL}_m(q)$ has subgroups $\mathrm{SL}_2(q)$ and $\mathrm{SL}_2(p)$; again Theorem 3 excludes all but finitely many primes p and, since also $\mathrm{SL}_2(q)$ has an elementary abelian subgroup $(\mathbb{Z}_p)^k$, by Theorem 4 only finitely many powers p^k of a fixed prime p can occur. Concluding, only finitely many prime powers p^k can occur for a fixed dimension n. Note that, in a similar way, also for the groups $\mathrm{SL}_2(q)$ only finitely many values of q can occur. We still have to bound m; if q is not a power of 2, then $\mathrm{PSL}_m(q)$ has a subgroup $(\mathbb{Z}_2)^{m-2}$ represented by diagonal matrices with entries ± 1 on the diagonal, so m is bounded by Theorem 4. If q is a power of 2, one may consider instead subgroups $(\mathbb{Z}_p)^{[m/2]} < \mathrm{SL}_2(p)^{[m/2]} < \mathrm{PSL}_m(q)$ and again apply Theorem 4.

This finishes the proof of Theorem 2 for the case of the projective linear groups $\mathrm{PSL}_m(q)$.

The proof for the *unitary groups* $\mathrm{PSU}_m(q)$ and the *symplectic groups* $\mathrm{PSp}_{2m}(q)$ is similar, noting that there are isomorphisms

$$\mathrm{PSU}_2(q) \cong \mathrm{PSp}_2(q) \cong \mathrm{PSL}_2(q), \;\; \mathrm{SU}_2(q) \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q);$$

in particular, if $m \geq 3$ or $2m \geq 4$, the latter groups are subgroups of both $\mathrm{PSU}_m(q)$ and $\mathrm{PSp}_{2m}(q)$ and we can conclude as before.

The last class of classical groups are the *orthogonal groups* $\Omega_{2m+1}(q) = P\Omega_{2m+1}(q)$ and $P\Omega^{\pm}_{2m}(q)$ (the latter stands for two different groups which are simple if $m \geq 3$). There are isomorphisms

$$\Omega_3(q) \cong \mathrm{PSL}_2(q), \; P\Omega^+_4(q) \cong \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q), \; P\Omega^-_4(q) \cong \mathrm{PSL}_2(q^2)$$

(see [Su, p.384]). By canonical inclusions between orthogonal groups and the cases considered before, this leaves again only finitely many possibilities.

Next we consider the *exceptional groups* $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, and also the Steinberg triality groups ${}^3D_4(q)$. By [St, Table 0A8], [GL, Table 4-1], up to central extensions there are inclusions

$$E_6(q) > F_4(q) > {}^3D_4(q) > G_2(q) > \mathrm{PSL}_3(q), \quad E_7(q) > \mathrm{PSL}_8(q), \quad E_8(q) > \mathrm{PSL}_9(q).$$

Applying Lemma 1 we reduce to subgroups $\mathrm{SL}_2(q)$ in all cases.

Finally, there are the *twisted groups*

$${}^2E_6(q), \;\; {}^3D_4(q), \;\; Sz(2^{2m+1}) = {}^2B_2(2^{2m+1}), \;\; {}^2G_2(3^{2m+1}), \;\; {}^2F_4(2^{2m+1}).$$

By [St, Table 0A8], [GL, Table 4-1], up to central extensions there are inclusions ${}^2E_6(q) > F_4(q)$ and ${}^3D_4(q) > G_2(q)$ (already considered before); the Suzuki groups $Sz(2^{2m+1})$ have subgroups $(\mathbb{Z}_2)^{2m+1}$ ([G1, p.74]), the Ree groups ${}^2G_2(3^{2m+1})$ subgroups $\mathrm{PSL}_2(3^{2m+1})$ ([G2, p.164]) and the Ree groups ${}^2F_4(2^{2m+1})$ subgroups $\mathrm{SU}_3(2^{2m+1})$ ([GL, Table 4-1]), so in all these cases some previously considered case applies.

This completes the second proof of Theorem 2.

The proof of Theorem 2 allows to produce for each dimension n a finite list of finite simple groups which are the candidates for actions on homology n-spheres; then one can identify those groups from the list which admit a linear action on S^n (or equivalently, have a faithful, real, linear representation in dimension $n+1$), and try to eliminate the remaining ones by refined methods. Along these lines, we proved the follwing for dimension 5.

Theorem 5. ([GZ]) *A finite simple group acting on a homology 5-sphere, and in particular on the 5-sphere, is isomorphic to one of the following groups:*
an alternating group $\mathbb{A}_5$, $\mathbb{A}_6$ *or* $\mathbb{A}_7$*; a linear fractional group* $\mathrm{PSL}_2(7)$*;*
a unitary group $\mathrm{PSU}_4(2)$ *or* $\mathrm{PSU}_3(3)$.

With the exception of the unitary group $\mathrm{PSU}_3(3)$ these are exactly the finite simple groups which admit a linear action on S^5. The alternating group $\mathbb{A}_7$ acts on $\mathbb{R}^7$ by permutation of coordinates, and hence also on a diagonal $\mathbb{R}^6$ and on S^5, and has the linear fractional group $\mathrm{PSL}_2(7)$ as a subgroup (since the latter contains the symmetric group $\mathbb{S}_4$ as a subgroup of index 7). The unitary group $\mathrm{PSU}_4(2)$, of order 25920, is a subgroup of index 2 in the Weyl group of type E_6 which has an integer linear representation in dimension 6 (this is one of the low-dimensional exceptions for the upper bound $j(n) \leq (n+1)!$ of the Jordan number). The group $\mathrm{PSU}_3(3)$ admits a linear action on S^6 (see e.g. [Con]); we conjecture that it does not act on a homology 5-sphere but are not able to exclude it at present.

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