The wave equation on hyperbolic spaces

Maria Vallarino

Dipartimento di Scienze Matematiche
Politecnico di Torino

Workshop su varieta' reali e complesse:
geometria, topologia e analisi armonica
Pisa, 28 febbraio – 3 marzo 2013

ABSTRACT. We study the dispersive properties of the wave equation associated with the shifted Laplace–Beltrami operator on real hyperbolic spaces, and deduce Strichartz estimates for a large family of admissible pairs. As an application, we obtain local well–posedness results for the nonlinear wave equation.

This is a joint work with J.-Ph. Anker and V. Pierfelice [2].

The euclidean wave equation

The study of the dispersive properties of the **linear wave equation on** $\mathbb{R}^n$

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_{\mathbb{R}^n} u(t,x) = F(t,x) \\ u(0,x) = f(x) \\ \partial_t|_{t=0}\, u(t,x) = g(x)\,, \end{cases} \tag{1}$$

is well established. In particular, the following **Strichartz estimates for solutions of the Cauchy problem** (1) **hold**:

$$\|u\|_{L^p(\mathbb{R};\,L^q)} + \|u\|_{L^\infty(\mathbb{R};\,\dot{H}^\sigma)} + \|\partial_t u\|_{L^\infty(\mathbb{R};\,\dot{H}^{\sigma-1})} \lesssim \|f\|_{\dot{H}^\sigma} + \|g\|_{\dot{H}^{\sigma-1}} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};\,\dot{H}^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'})}\,, \tag{2}$$

under the assumptions that

$$\sigma \geq \tfrac{n+1}{2}\left(\tfrac{1}{2}-\tfrac{1}{q}\right), \quad \tilde{\sigma} \geq \tfrac{n+1}{2}\left(\tfrac{1}{2}-\tfrac{1}{\tilde{q}}\right),$$

and the couples $(p,q), (\tilde{p},\tilde{q}) \in (2,\infty] \times [2, 2\,\frac{n-1}{n-3})$ satisfy

$$\tfrac{2}{p} + \tfrac{n-1}{q} = \tfrac{n-1}{2}, \quad \tfrac{2}{\tilde{p}} + \tfrac{n-1}{\tilde{q}} = \tfrac{n-1}{2}. \tag{3}$$

The couples $(p,q),(\tilde{p},\tilde{q})$ **for which** (3) **hold are called** $\mathbb{R}^n$**–admissible**. The estimate (2) holds also at the endpoint $(2, 2\,\frac{n-1}{n-3})$ when $n \geq 4$. When $n = 3$ this endpoint is $(2,\infty)$ and the estimate (2) fails ([5, 7]).

The previous estimates yield existence results for the nonlinear wave equation in the euclidean setting. The problem of finding minimal regularity on initial data ensuring local well–posedness for semilinear wave equation was almost completely answered in [4, 7].

Once the euclidean case was more or less settled, several attemps have been made in order to establish Strichartz estimates for dispersive equations in other settings. Here we consider real hyperbolic spaces $\mathbb{H}^n$**, which are the most simple examples of noncompact Riemannian manifolds with negative curvature.**

The shifted wave equation on hyperbolic spaces

We denote by $\mathbb{H}^n$ the **real hyperbolic space of dimension** $n \geq 3$

$$\mathbb{H}^n = \{x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1},\; x_0 \geq 1, x_0^2 - x_1^2 - \dots - x_n^2 = 1, \}$$

equipped with the Riemannian metric

$$ds^2 = -dx_0^2 + dx_1^2 + \dots + dx_n^2\,.$$

In geodesic polar coordinates the Riemannian metric is given by

$$ds^2 = dr^2 + \sinh^2 r\, d\omega^2,$$

the Riemannian volume by

$$dv = (\sinh r)^{n-1} dr\, d\omega_{\mathbb{S}^{n-1}}\,,$$

and the Laplace-Beltrami operator by

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}.$$

The spectrum of $-\Delta_{\mathbb{H}^n}$ on $L^2(\mathbb{H}^n)$ is $[\rho^2, +\infty)$, where $\rho = \frac{n-1}{2}$.

We study the **linear shifted wave equation**

$$\begin{cases} \partial_t^2 u(t,x) - (\Delta_{\mathbb{H}^n} + \rho^2)\, u(t,x) = F(t,x) \\ u(0,x) = f(x) \\ \partial_t|_{t=0}\, u(t,x) = g(x)\,, \end{cases} \tag{4}$$

whose solution is given by Duhamel's formula :

$$u(t,x) = (\cos tD)\, f(x) + \tfrac{\sin tD}{D}\, g(x) + \int_0^t ds\, \tfrac{\sin(t-s)D}{D}\, F(s,x)\,,$$

where $D = \sqrt{-\Delta_{\mathbb{H}^n} - \rho^2}$.

This equation can be considered as an analog of the euclidean wave equation in the hyperbolic setting, since the bottom of the L^2-spectrum of $-\Delta_{\mathbb{H}^n} - \rho^2$ is 0. The shifted wave equation (4) was previously considered by D. Tataru [9] and A. Ionescu [6].

A couple (p,q) is called $\mathbb{H}^n$**–admissible** if $(\frac{1}{p},\frac{1}{q})$ belongs to the triangle

$$T_n = \left\{ \left(\tfrac{1}{p},\tfrac{1}{q}\right) \in \left(0,\tfrac{1}{2}\right) \times \left(0,\tfrac{1}{2}\right) \;\middle|\; \tfrac{2}{p} + \tfrac{n-1}{q} \geq \tfrac{n-1}{2} \right\}, \tag{5}$$

together with the endpoint $\left(\frac{1}{2},\frac{n-3}{2(n-1)}\right)$ in dimension $n \geq 4$.

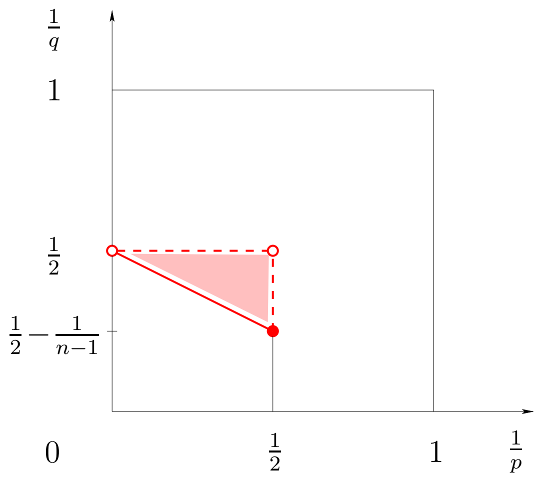

Figure 1: Admissibility in dimension $n \geq 4$

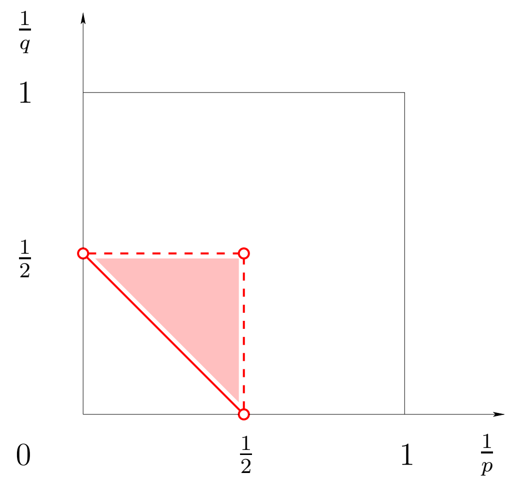

Figure 2: Admissibility in dimension $n=3$

Notice that the set of $\mathbb{H}^n$-admissible couples is larger than the set of $\mathbb{R}^n$-admissible couples: this follows from stronger dispersive properties of the hyperbolic wave propagator with respect to the euclidean one.

Theorem (Anker-Pierfelice-V.) Let (p,q) and $(\tilde{p},\tilde{q})$ be two $\mathbb{H}^n$–admissible couples. Then the following Strichartz estimates hold for solutions to the Cauchy problem (4):

$$\|u\|_{L^p(\mathbb{R};L^q)} \lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + \|F\|_{L^{\tilde{p}'}\left(\mathbb{R};H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'}\right)}, \tag{6}$$

where $\sigma \geq \frac{(n+1)}{2}\left(\frac{1}{2}-\frac{1}{q}\right)$ and $\tilde{\sigma} \geq \frac{(n+1)}{2}\left(\frac{1}{2}-\frac{1}{\tilde{q}}\right)$. Moreover,

$$\begin{aligned} &\|u\|_{L^\infty\left(\mathbb{R};H^{\sigma-\frac{1}{2},\frac{1}{2}}\right)} + \|\partial_t u\|_{L^\infty\left(\mathbb{R};H^{\sigma-\frac{1}{2},-\frac{1}{2}}\right)} \\ &\lesssim \|f\|_{H^{\sigma-\frac{1}{2},\frac{1}{2}}} + \|g\|_{H^{\sigma-\frac{1}{2},-\frac{1}{2}}} + \|F\|_{L^{\tilde{p}'}\left(\mathbb{R};H^{\sigma+\tilde{\sigma}-1}_{\tilde{q}'}\right)}. \end{aligned} \tag{7}$$

The Sobolev spaces involved in the Strichartz estimates above are suitable spaces naturally related to the conservation laws of the shifted wave equation defined as

$$H^s_q = (-\Delta_{\mathbb{H}^n})^{-s/2}(L^q(\mathbb{H}^n)) \qquad \text{and} \qquad H^{s,\tau} = (-\Delta_{H^n}+1)^{-s/2}\,(-\Delta_{H^n}-\rho^2)^{-\tau/2}(L^2(\mathbb{H}^n)),$$

for every $q\in(1,\infty)$, $s\in\mathbb{R}$, $\tau<3/2$.

The Strichartz estimates above are obtained using the spherical Fourier analysis on the hyperbolic space.

LWP results for NLW equation on $\mathbb{H}^n$

We shall first assume $n\geq 4$. We apply Strichartz estimates for the inhomogeneous linear Cauchy problem associated with the wave equation to prove local well–posedness results for the following **nonlinear Cauchy problem**

$$\begin{cases} \partial_t^2 u(t,x) - (\Delta_{\mathbb{H}^n}+\rho^2)\,u(t,x) = F(u(t,x)) \\ u(0,x) = f(x) \\ \partial_t|_{t=0}\,u(t,x) = g(x)\,, \end{cases} \tag{8}$$

with a **power–like nonlinearity** $F(u)$ such that

$$|F(u)| \leq C\,|u|^\gamma \quad \text{and} \quad |F(u)-F(v)| \leq C\,(|u|^{\gamma-1}+|v|^{\gamma-1})\,|u-v| \tag{9}$$

for some $C\geq 0$ and $\gamma>1$. The amount of smoothness σ requested for local well-posedness of (8) in $H^{\sigma-\frac{1}{2},\frac{1}{2}}\times H^{\sigma-\frac{1}{2},-\frac{1}{2}}$ depends on the nonlinearity power γ and is described in the following result.

Theorem (Anker-Pierfelice-V.) Let $n \geq 4$ and assume that $F(u)$ satisfies (9). Then the **NLW** (8) **is locally well–posed** in $H^{\sigma-\frac{1}{2},\frac{1}{2}} \times H^{\sigma-\frac{1}{2},-\frac{1}{2}}$ in the following cases:

(A) $1 < \gamma \leq \gamma_1$ and $\sigma > 0$;

(B) $\gamma_1 < \gamma \leq \gamma_2$ and $\sigma \geq C_1(\gamma)$;

(C) $\gamma_2 \leq \gamma < \gamma_{\text{conf}}$ and $\sigma \geq C_2(\gamma)$;

(D) $\gamma_{\text{conf}} \leq \gamma < \gamma_\infty$ and $\sigma > C_3(\gamma)$,

where

$$\gamma_1 = \tfrac{n+3}{n} = 1 + \tfrac{3}{n}, \qquad \gamma_2 = \tfrac{(n+1)^2}{(n-1)^2+4} = 1 + \tfrac{2}{\frac{n-1}{2} + \frac{2}{n-1}}, \qquad \gamma_{\text{conf}} = \tfrac{n+3}{n-1} = 1 + \tfrac{4}{n-1},$$

$$\gamma_3 = \tfrac{n^2+5n-2+\sqrt{n^4+2n^3+21n^2-12n+4}}{2n^2-2n}, \; \gamma_4 = \tfrac{n^2+2n-5}{n^2-2n-1}, \; \gamma_\infty = \min\{\gamma_3, \gamma_4\} = \begin{cases} \gamma_3 & \text{if } n = 4, 5 \\ \gamma_4 & \text{if } n \geq 6 \end{cases}$$

and the curves C_1, C_2, C_3 are given by

$$C_1(\gamma) = \tfrac{n+1}{4}\left(1 - \tfrac{n+5}{2n\gamma - n - 1}\right), \qquad C_2(\gamma) = \tfrac{n+1}{4} - \tfrac{1}{\gamma-1}, \qquad C_3(\gamma) = \tfrac{n}{2} - \tfrac{2}{\gamma-1}.$$

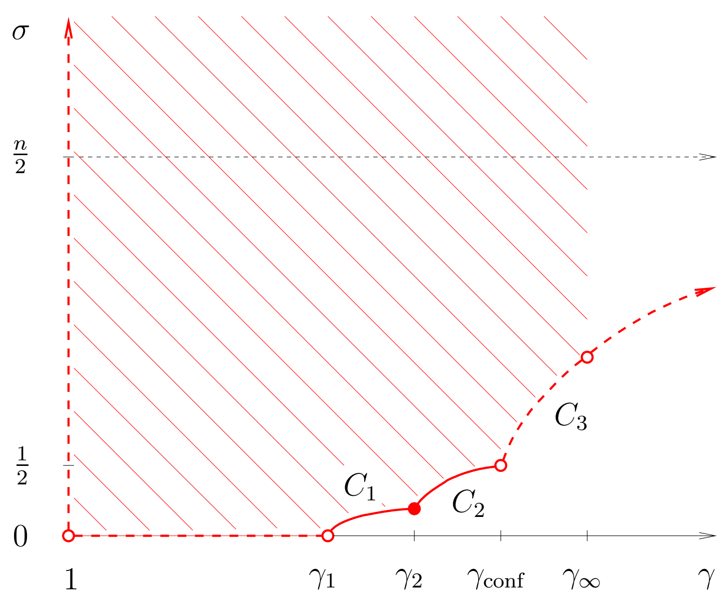

Figure 3: Regularity in dimension $n \geq 4$

Remark In dimension $n=3$, the Strichartz estimates are available in the triangle T_3 without the endpoint. The NLW (8) is locally well–posed in $H^{\sigma-\frac{1}{2},\frac{1}{2}} \times H^{\sigma-\frac{1}{2},-\frac{1}{2}}$ if

- $1<\gamma\le\gamma_1=2$ and $\sigma>0$;
- $2<\gamma<\gamma_{\text{conf}}=3$ and $\sigma\ge C_2(\gamma)=1-\frac{1}{\gamma-1}$;
- $3\le\gamma<\gamma_3=\frac{11+\sqrt{73}}{6}$ and $\sigma>C_3(\gamma)=\frac{3}{2}-\frac{2}{\gamma-1}$.

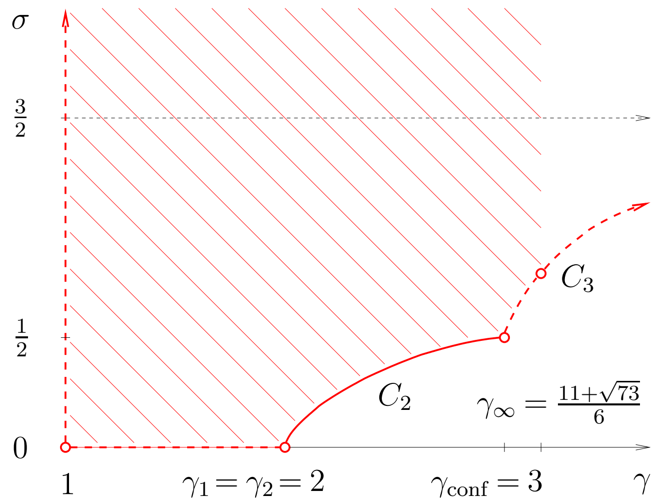

Figure 4: Regularity in dimension $n=3$

m

Further comments

- We also studied the shifted wave equation on Damek–Ricci spaces, a class of harmonic manifolds which are nonsymmetric in general and which contains hyperbolic spaces as a subclass. On Damek–Ricci spaces we prove Strichartz estimates for the linear shifted wave equation and obtain global well-posedness results for power like nonlinearities [3].
- Anker and Pierfelice [1], J. Metclafe and M. Taylor [8] investigated the wave equation associated with the nonshifted wave equation on hyperbolic spaces.
- It should be interesting to study dispersive properties of the shifted and nonshifted wave equations on noncompact symmetric spaces of arbitrary rank: the spherical analysis in this setting is more delicate and it would require a hard work.

References

[1] J.-Ph. Anker, V. Pierfelice, *Wave and Klein-Gordon equations on hyperbolic spaces*, arXiv:1104.0177

[2] J.-Ph. Anker, V. Pierfelice, M. Vallarino, *The wave equation on hyperbolic spaces*, J. Differential Equations 252 (2012), 5613–5661

[3] J.-Ph. Anker, V. Pierfelice, M. Vallarino, *The wave equation on Damek-Ricci spaces*, arXiv:1012.0689

[4] V. Georgiev, H. Lindblad, C. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. 119 (1997), 1291–1319

[5] J. Ginibre, G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. 133 (1995), 50–68

[6] A.D. Ionescu, *Fourier integral operators on noncompact symmetric spaces of real rank one*, J. Funct. Anal. 174 (2000), 274–300

[7] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), 955–980

[8] J. Metcalfe, M. Taylor, *Nonlinear waves on 3D hyperbolic space*, Trans. Amer. Math. Soc. 363 (2011), 3489–3529

[9] D. Tataru, *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*, Trans. Amer. Math. Soc. 353 (2001), 795–807