Geometria Simplettica e metriche Hermitiane speciali

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1 Marzo 2013

Tamed almost complex structures

Definition

An almost cx structure J on a symplectic manifold (M^{2n}, Ω) is tamed by Ω if $\Omega(X, JX) > 0$, $\forall X \neq 0$.

If J is tamed by Ω, then

$$g(X, Y) = \frac{1}{2}(\Omega(X, JY) - \Omega(JX, Y))$$

is a J-Hermitian metric.

Theorem (Streets, Tian; Li, Zhang)

If a compact complex (M^4, J) admits a symplectic structure taming J, then (M^4, J) has a Kähler metric.

Problem

Does there it exist an example of a compact complex (M^{2n}, J), with $n > 2$, admitting a symplectic form taming J, but no Kähler structures?

We will give some negative answer to this problem.

Link with SKT metrics

• Giving a symplectic form Ω taming a complex structure J on M^{2n} is equivalent to assign a J-Hermitian g whose fundamental 2-form $F = g(J\cdot, \cdot)$ satisfies $\partial F = \overline{\partial}\beta$, for some ∂-closed $(2,0)$-form β.

Thus in particular $\partial\overline{\partial}F = 0$.

Definition

A J-Hermitian metric g on a complex manifold (M^{2n}, J) is said to be strong Kähler with torsion (SKT) or pluriclosed if $\partial\overline{\partial}F = 0$.

Bismut connection

Proposition (Bismut)

On any Hermitian (M^{2n}, J, g) $\exists!$ connection ∇^B such that

$$\begin{array}{ll} \nabla^B g = 0 & \text{(metric)} \\ \nabla^B J = 0 & \text{(Hermitian)} \\ c(X,Y,Z) = g(X, T^B(Y,Z)) & \text{3-form} \end{array}$$

where T^B is the torsion of ∇^B.

$\nabla^B = \nabla^{LC} + \frac{1}{2} c$ is the Bismut connection and $c = -JdJF$.

$c = 0 \Longleftrightarrow \nabla^B = \nabla^{LC} \Longleftrightarrow (M^{2n}, J, g)$ is Kähler

(J, g) on M^{2n} is SKT if and only if $dc = 0$.

The pluriclosed flow

Let (M^{2n}, J, g_0) be an Hermitian manifold. Streets and Tian introduced the pluriclosed flow

$$\begin{cases} \frac{\partial F(t)}{\partial t} = \Phi(F(t)), \\ F(0) = F_0 \end{cases},$$

where $\Phi(F) = -\partial\partial^* F - \overline{\partial\partial}^* F - \frac{i}{2}\partial\overline{\partial} \log \det g = -(\rho^B)^{1,1}(F)$.

Proposition (Streets, Tian)

Let (M^{2n}, J, g) be a SKT manifold. Then $F \to \Phi(F)$ is a real quasi-linear second-order elliptic operator when restricted to $\{J - \text{Hermitian SKT metrics}\}$.

If $g(0)$ is SKT (Kähler), then $g(t)$ is SKT (Kähler) for all t.

Definition (Streets, Tian)

A SKT metric g on compact (M^{2n}, J) is static if $\Phi(F) = \lambda F$, or equivalently if $(\rho^B)^{1,1} = \lambda F$, for a constant λ.

If g is Kähler and static, then it is Kähler-Einstein.

Proposition (Streets, Tian)

Let (M^{2n}, J) be compact with a static SKT metric g. If $\lambda \neq 0$, then $F = \Omega^{1,1}$ with Ω is a symplectic form Ω taming J.

Given a symplectic form Ω_0 taming a complex structure J on M^{2n} we can define the HS flow

$$\begin{cases} \frac{\partial \Omega(t)}{\partial t} = -\rho^B(\Omega^{1,1}(t)) \\ \Omega(0) = \Omega_0. \end{cases}$$

• We have a short-time existence of solutions and $\Omega(t)$ is taming J for every t.

Problem

Study the properties of previous HS flow.

Still in progress!

Results on nilmanifolds

Theorem (Enrietti, -, Vezzoni)

$(G/\Gamma, J)$ *with* J *left-invariant and* G *any Lie group*
If $J\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$, *then* $(G/\Gamma, J)$ *does not admit any compatible Hermitian-symplectic structure.*

Theorem (Enrietti, -, Vezzoni)

$M^{2n} = G/\Gamma$ *nilmanifold (not a torus),* J *left-invariant.*
1) *If* (M^{2n}, J) *has a SKT metric, then* G *has to be 2-step and* (M^{2n}, J) *is a principal holomorhic torus bundle over a torus.*
2) (M^{2n}, J) *does not admit any symplectic form taming* J.

Remark

2) was proved for $n = 3$ by Angella and Tomassini.

To prove 1) we show that J has to preserve the center ξ of $\mathfrak{g}$ and that a SKT structure on $\mathfrak{g}$ induces a SKT structure on $\mathfrak{g}/\xi$.

To prove 2) we may assume that the Hermitian-symplectic form is invariant using the symmetrization process.

Since J preserves the center ξ and $[\mathfrak{g}, \mathfrak{g}] \subseteq \xi$, we have that $J\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$ and so $(\mathfrak{g}, J)$ does not admit any symplectic form taming J.

The Hermitian flow on nilmanifolds

Theorem (Enrietti, -, Vezzoni)

Let $(G/\Gamma, J)$ be a complex nilmanifold with an invariant SKT structure F_0. Then the pluriclosed flow

$$\begin{cases} \frac{\partial F(t)}{\partial t} = -(\rho^B)^{1,1}(F(t)), \\ F(0) = F_0, \end{cases}$$

has a unique solution $F(t)$ defined for every $t \in \mathbb{R}^+$.

To prove the theorem we use that the pluriclosed flow on a SKT Lie algebra can be studied using the Lauret trick and deforming the Lie bracket instead of the Hermitian metric.

Let $(\mathbb{R}^{2n}, J_0, <\,,\,>)$ and

$$\mathcal{N}_2 = \{\mu \in \Lambda^2 \otimes \Lambda^1 \mid \mu(\mu(\cdot,\cdot),\cdot) = 0, N_\mu = 0\}.$$

$$\begin{cases} \frac{\partial F(t)}{\partial t} = (dd^* F)^{1,1}(t), \\ F(0) = F_0, \end{cases} \Longleftrightarrow \begin{cases} \frac{\partial \mu(t)}{\partial t} = \frac{1}{2}\delta_\mu P_\mu(t), \\ \mu(0) = \mu_0, \end{cases}$$

where

$$\delta_\mu P_\mu = \mu(P_\mu \cdot, \cdot) + \mu(\cdot, P_\mu \cdot) - P_\mu \mu(\cdot,\cdot), \;< P_\mu(Z_i), Z_{\bar{j}} >= \rho^B(Z_i, Z_{\bar{j}}).$$

We have

$$< \delta_\mu P_\mu, \mu >= -8 < P_\mu, P_\mu > \leq 0 \Rightarrow \frac{\partial}{\partial t} < \mu(t), \mu(t) > \leq 0.$$

Calibrated almost complex structures

Definition

An almost cx structure J on a symplectic manifold (M^{2n}, Ω) is calibrated by Ω (or Ω is compatible with J) if J is tamed and $\Omega(JX, JY) = \Omega(X, Y)$, $\forall X, Y$.

• If J is calibrated by $\Omega \Longrightarrow (\Omega, J)$ is an almost-Kähler (AK) structure $\Rightarrow g(X, Y) = \Omega(X, JY)$ is a J-Hermitian metric.

• If J is integrable, then the AK structure (Ω, J, g) is Kähler.

The Symplectic Calabi-Yau problem

Theorem (Yau, Symplectic version)

Let (M^{2n}, J, Ω) be a compact Kähler manifold and let σ be a volume form satisfying $\int_{M^{2n}} \Omega^n = \int_{M^{2n}} \sigma$. Then there exists a unique Kähler form $\tilde{\Omega}$ with $[\tilde{\Omega}] = [\Omega]$ such that

$$\tilde{\Omega}^n = \sigma \longleftrightarrow \text{CY Equation}$$

CY equation $\leftrightarrow \begin{cases} (\Omega + d\alpha)^n = e^f \Omega^n, \\ J(d\alpha) = d\alpha \end{cases} \leftrightarrow (\Omega + dd^c h)^n = e^f \Omega^n.$

$(\Omega + dd^c h)^n = e^f \Omega^n$ is a Monge-Ampère equation in h.

$$\text{CY equation} \leftrightarrow \begin{cases} (\Omega + dd^c h)^n = e^f \Omega^n, \\ \int_{M^{2n}} h\, \Omega^n = 0. \end{cases} \qquad (*)$$

Yau's theorem: $(*)$ has always a unique solution h.

Donaldson introduced the symplectic version of the CY equation for AK manifolds.

Symplectic CY problem

Let (M^{2n}, J, Ω, g) be a compact AK manifold with a volume form $\sigma = \mathrm{e}^f \Omega^n$ satisfying $\int_{M^{2n}} \mathrm{e}^f \Omega^n = \int_{M^{2n}} \Omega^n$. Then

$$\text{CY equation} \longleftrightarrow \begin{cases} (\Omega + d\alpha)^n = e^f \Omega^n \\ J(d\alpha) = d\alpha \end{cases} \qquad (*)$$

- $(*)$ is elliptic for $n = 2$ and the solutions are unique [Donaldson];
- $(*)$ is overdetermined for $n > 2$.

Problem

Can the Yau Theorem be generalized to AK 4-manifolds?

On a AK (M^4, Ω, J, g) $\exists!$ connection ∇^C (the canonical or Chern connection) such that $\nabla^C J = \nabla^C \Omega = 0\,,\quad \mathrm{Tor}^{1,1}(\nabla^C) = 0\,.$

Theorem (Tosatti, Weinkove, Yau)

If $\mathcal{R}(g, J) \geq 0$, then the Calabi-Yau problem can be solved for every normalized volume form on (M^4, Ω, J, g), where $\mathcal{R}(g, J)$ is defined by $\mathcal{R}_{i\bar{j}k\bar{l}} = R^{j}_{ik\bar{l}} + 4N^{r}_{\bar{l}\bar{j}}\overline{N^{i}_{\bar{r}k}}$.

- The theorem can be applied to an infinitesimal deformation of the Fubini-Study Kähler structure on $\mathbb{CP}^n$ but it cannot be applied for instance to the Kodaira-Thurston surface!
- We will study the CY problem on T^2-bundles over $\mathbb{T}^2$.

The CY equation on the Kodaira-Thurston manifold

Remark

The existence result by Tosatti - Weinkove-Yau cannot be applied to the KT manifold $M^4 = (\Gamma \backslash Nil^3) \times S^1$.

$$Nil^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\}$$

- M^4 has a global invariant coframe $\{e^i\}$ such that

$$e^1 = dy, \qquad e^2 = dx, \qquad e^3 = dt, \qquad e^4 = dz - x\,dy.$$

We will denote simply $Nil^3 \times \mathbb{R}$ by $(0, 0, 0, 12)$.

- M^4 is the total space of a T^2-bundle over $\mathbb{T}^2$:

$$T^2 = S^1 \times S^1 \hookrightarrow \Gamma\backslash Nil^3 \times S^1$$
$$\downarrow \pi_{xy}$$
$$\mathbb{T}^2_{xy}$$

- M^4 has the Lagrangian (with respect to π_{xy}) AK structure

$$\Omega = e^1 \wedge e^4 + e^2 \wedge e^3, \quad g = \sum_{i=1}^{4} e^i \otimes e^i \,,$$

i.e. Ω vanishes on the fibers.

Theorem (Tosatti,Weinkove)

The CY equation on the KT manifold (M^4, J, Ω, g) can be solved for every T^2-invariant volume form σ.

Let $\alpha = v\, e^1 + v_x\, e^3 + v_y\, e^4$, $\quad v \in C^\infty(\mathbb{T}^2)$.

Then $d\alpha = v_{xx}\, e^{23} + v_{xy}(e^{13} + e^{24}) + v_{yy}\, e^{14}$ and the CY equation $(\Omega + d\alpha)^2 = e^f\, \Omega^2$ becomes the Monge-Ampère equation

$$(1 + v_{xx})(1 + v_{yy}) - v_{xy}^2 = e^f$$

Theorem (Li)

The Monge-Ampère equation on the standard torus T^n has always a solution.

Goal: To generalize this argument to other AK structures on T^2-bundles over $\mathbb{T}^2$.

Definition (Thurston)

A geometric 4-manifold is a pair (X, G) where X is a complete, simply-connected Riemannian 4-manifold, G is a group of isometries acting transitively on X that contains a discrete subgroup Γ such that $\Gamma \backslash X$ has finite volume.

Let $Nil^4 = (0, 13, 0, 12), \quad Sol^3 \times \mathbb{R} = (0, 0, 13, 41)$.

Theorem (Ue)

Every orientable T^2-bundle over a $\mathbb{T}^2$ is a geometric 4-manifold, where (X, G) is one of the following

$$(\mathbb{R}^4, SO(4) \ltimes \mathbb{R}^4), \quad (Nil^3 \times \mathbb{R}, Nil^3 \times S^1),$$
$$(Nil^4, Nil^4), \quad (Sol^3 \times \mathbb{R}, Sol^3 \times \mathbb{R})$$

and it is infra-solvmanifold, i.e. a smooth quotient $\Gamma \backslash X$ covered by a solvmanifold or equivalently a quotient $\Gamma \backslash X$, where the discrete group Γ contains a lattice $\tilde{\Gamma}$ of X such that $\tilde{\Gamma} \backslash \Gamma$ is finite.

Definition

An AK structure (J, Ω, g) on an infra-solvmanifold $M^4 = \Gamma \backslash X$ is called invariant if it induced by a left-invariant on X and it is Γ-invariant.

Proposition (-, Li, Salamon, Vezzoni)

On a 4-dimensional infra-solvmanifold $(M^4 = \Gamma \backslash X, J, \Omega, g)$ with an invariant AK structure, the Tosatti-Weinkove-Yau condition $\mathcal{R}(g, J) \geq 0$ is satisfied if and only if (Ω, J) is Kähler.

The main result

Theorem (–, Li, Salamon, Vezzoni / Buzano, –, Vezzoni)

Let $M^4 = \Gamma\backslash X$ be a T^2-bundle over a $\mathbb{T}^2$ endowed with an invariant AK structure (J, Ω, g). Then for every normalized T^2-invariant volume form $\sigma = e^F\,\Omega^2$, $F \in C^\infty(\mathbb{T}^2)$ the associated CY problem has a unique solution.

Layout of the proof:

- Use the classification of T^2-bundles of $\mathbb{T}^2$;
- Classify in each case invariant Lagrangian AK structures and invariant non-Lagrangian AK structures;
- Rewrite the problem in terms of a Monge-Ampère equation;
- Show that such an equation has a solution.

Classification of T^2-bundles over $\mathbb{T}^2$

By Sakamoto and Fukuhara the diffeomorphic classes of T^2-bundles over $\mathbb{T}^2$ are classified in 8 families:

	G	Structure equations of X
$i), ii)$	$SO(4) \ltimes \mathbb{R}^4$	$(0,0,0,0)$
$iii), iv), v)$	$Nil^3 \times S^1$	$(0,0,0,12)$
$vi)$	Nil^4	$(0,13,0,12)$
$vii), viii)$	$Sol^3 \times \mathbb{R}$	$(0,0,13,41)$

The Lie group G is the geometry type of $\Gamma \backslash X$.

• In the cases different from $iii)$ the fibration of M^4 as torus bundle is unique.

• In the case $iii)$ one has two fibrations

$$\pi_{xy} : M^4 \longrightarrow \mathbb{T}^2_{xy}, \quad \pi_{yt} : M^4 \longrightarrow \mathbb{T}^2_{yt}.$$

Theorem (Geiges)

Let $M^4 = \Gamma \backslash X$ be an orientable T^2-bundle over a $\mathbb{T}^2$. Then

- *M^4 has a symplectic form and every class $a \in H^2(M^4, \mathbb{R})$ with $a^2 \neq 0$ can be represented by a symplectic form;*
- *M^4 has a Kähler structure if and only if $X = \mathbb{R}^4$;*
- *If $X = Nil^4$ then every invariant AK structure on M^4 is Lagrangian;*
- *If $X = Sol^3 \times \mathbb{R}$ every invariant AK structure on M^4 is non-Lagrangian.*

Classification of invariant AK structures

Goal: Classify all invariant AK structures (g, Ω) on $Nil^3 \times \mathbb{R}$, Nil^4, $Sol^3 \times \mathbb{R}$.

In each case there exists an ON basis (f^i) such that $\Omega = f^{12} + f^{34}$ and

- $G = Nil^4 \to f^1 \in < e^1 >$, $f^2 \in < e^1, e^2 >$, $f^3 \in < e^1, e^2, e^3 >$.
- $G = Sol^3 \times \mathbb{R} \to f^1 \in < e^1 >$, $f^3 \in < e^3 >$, $f^4 \in < e^3, e^4 >$.
- $G = Nil^4 \to f^1 \in < e^1 >$, $g(e^3, f^2) = 0$, $g(e^3, f^3) g(e^4, f^4) \geq 0$.

- The case $X = Nil^3 \times \mathbb{R}$

	G	Structure equations of X
$i), ii)$	$SO(4) \ltimes \mathbb{R}^4$	$(0,0,0,0)$
$iii), iv), v)$	$Nil^3 \times S^1$	$(0,0,0,12)$
$vi)$	Nil^4	$(0,13,0,12)$
$vii), viii)$	$Sol^3 \times \mathbb{R}$	$(0,0,13,41)$

- In this case all the total spaces M^4 are nilmanifolds.
- All the invariant AK structures are Lagrangian and we can work as for the Kodaira-Thurston surface.

- The case $X = Nil^3 \times \mathbb{R}$

	G	Structure equations of X
$i), ii)$	$SO(4) \ltimes \mathbb{R}^4$	$(0,0,0,0)$
$iii), iv), v)$	$Nil^3 \times S^1$	$(0,0,0,12)$
$vi)$	Nil^4	$(0,13,0,12)$
$vii), viii)$	$Sol^3 \times \mathbb{R}$	$(0,0,13,41)$

- The total space M^4 could be also a infra-nilmanifold.
- The invariant AK structures on M^4 could be either Lagrangian or non-Lagrangian and the argument used for the Kodaira-Thurston surface has to be modified.

- The case $X = Sol^3 \times \mathbb{R}$

	G	Structure equations of X
$i), ii)$	$SO(4) \ltimes \mathbb{R}^4$	$(0, 0, 0, 0)$
$iii), iv), v)$	$Nil^3 \times \mathbb{R}$	$(0, 0, 0, 12)$
$vi)$	Nil^4	$(0, 13, 0, 12)$
$vii), viii)$	$Sol^3 \times \mathbb{R}$	$(0, 0, 13, 41)$

In this case the total space could be an infra-sovmanifold, all invariant AK structures are non-Lagrangian and the CY equation reduces to a Monge-Ampère equation.

- The case $X = Nil^4$

	G	Structure equations of X
$i), ii)$	$SO(4) \ltimes \mathbb{R}^4$	$(0,0,0,0)$
$iii), iv), v)$	$Nil^3 \times \mathbb{R}$	$(0,0,0,12)$
$vi)$	Nil^4	$(0,13,0,12)$
$vii), viii)$	$Sol^3 \times \mathbb{R}$	$(0,0,13,41)$

In this case the total spaces are nilmanifolds, all the invariant AK structure are Lagrangian and the CY equation reduces to the same Monge-Ampère equation for Lagrangian AK structures in the families iv) and v) associated to $Nil^3 \times \mathbb{R}$.

The Monge-Ampère equation

The following equation covers all cases

$$A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + E_2 e^f,$$

where

$$\begin{aligned} A_{11}[u] &= u_{xx} + B_{11} u_x + C_{11} + Du, \\ A_{12}[u] &= u_{xy} + B_{12} u_y + C_{12}, \\ A_{22}[u] &= u_{yy} + B_{22} u_y + C_{22} \end{aligned}$$

and B_{ij}, C_{ij}, D, E_i are constants.

In the Lagrangian case $D = 0$.

Solutions to the Monge-Ampère equation

Goal: Show that $A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + E_2 e^f$ has a solution on $\mathbb{T}^2$.

• The first step consists to show that the solutions to the equation are unique up to a constant.

• We look for a solution u satisfying $\int_{\mathbb{T}^2} u = 0$.

• We apply the continuity method to

$$A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-t)E_2 + tE_2 e^f, \quad t \in [0,1].$$

using a priori estimate

$$\|u\|_{\mathcal{C}^2} \leq 2(B_{11}+1)|B_{22}|e^{2C_{22}} + C_{11} + C_{22}.$$

Open related problems

• Find a (generalized) $\partial\overline{\partial}$-lemma which ensures that the symplectic CY problem reduces to a Monge-Ampère equation.

• Find a proof of the main theorem in terms of a (modified) Ricci flow.

• Find examples of compact AK (non Kähler) manifolds with $\mathcal{R}(g, J) > 0$.

• Introduce and study a CY equation for other geometric structures (positive 3-forms in dimension 6, G_2-structures with torsion in dimension 7, $Spin(7)$-structures with torsion in dimension 8...).

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GRAZIE PER L'ATTENZIONE!!