Singular integrals on Riemannian manifolds

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Singular integrals in $\mathbb{R}^n$

Integral operators

$$Tf(x) = p.v. \int K(x,y)\, f(y)\, \mathrm{d}y$$

with a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which has a critical singularity at 0 and ∞:

$$K(x,y) \simeq |x-y|^{-n}, \qquad \nabla_{x,y} K(x,y) \simeq |x-y|^{-n-1}$$

as $|x-y| \to 0,\ \infty$.

The integral must be interpreted as a principal value:

$$Tf(x) = \lim_{\epsilon \to 0} \int_{\epsilon < |x-y| < \epsilon^{-1}} K(x,y)\, f(y)\, \mathrm{d}y$$

These kernels are called Calderón-Zygmund kernels.

The main result of the C-Z theory

THEOREM *If T is bounded on $L^2(\mathbb{R}^n)$ and the kernel K is Calderón-Zygmund then*

- *T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$;*
- *T is of weak type $(1,1)$ i.e.*

$$\sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq C\|f\|_1.$$

Key ingredient of the proof of the weak-type (1,1):
the Calderón-Zygmund decomposition.

It relies on the doubling property of the Lebesgue measure:

$$V(x,2r) \leq C_n\, V(x,r) \qquad \forall x \in \mathbb{R}^n, r > 0.$$

Here $V(x,r) = m(B(x,r))$.

APPLICATIONS:

- convergence of Fourier series in L^p
- regularity of solutions of elliptic and parabolic PDE's
- representations of Lie groups
- generalizations of the Cauchy integral to several complex variables

TWO EXAMPLES

- Spectral multipliers of the Laplacian
- Riesz transforms

Spectral multipliers

$\Delta = -\sum_{j=1}^{n} \partial_j^2$, $\mathcal{F} =$ Fourier transform

$$\Delta = \int_0^\infty \lambda \, \mathrm{d}P_\lambda, \qquad P_\lambda f = \mathcal{F}^{-1} \chi_{B(0,\sqrt{\lambda})} \mathcal{F} f$$

If $m : \mathbb{R}_+ \to \mathbb{C}$, bounded, define

$$m(\Delta) = \int_0^\infty m(\lambda) \, \mathrm{d}P_\lambda.$$

If $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and

$$|\partial^\alpha m(\xi)| \le C_\alpha \, |\xi|^{-|\alpha|}, \qquad \forall \xi \neq 0, \ \forall \alpha \in \mathbb{N}^n$$

then $m(\Delta)$ is Calderón-Zygmund. Proof via Fourier analysis because

$$K_m(x,y) = \mathcal{F}^{-1} m(|\bullet|^2)(x-y).$$

Riesz transforms

The vector valued Riesz transform is the operator $\nabla \Delta^{-1/2}$.

- It is bounded on $L^2(\mathbb{R}^n)$ (Fourier analysis)
- It is a C-Z operator because

$$\nabla \Delta^{-1/2} f(x) = \mathrm{p.v.} \int K(x-y)\, f(y) \mathrm{d}y$$

where

$$K(x-y) = \mathcal{F}^{-1}\left(\frac{i\xi}{|\xi|}\right)(x-y) = c_n \; \frac{x-y}{|x-y|^{n+1}}$$

S. I. on Riemannian manifolds

M complete non-compact connected Riemannian manifold

- ρ geodesic distance
- μ Riemannian measure ($\mu(M) = \infty$)
- d exterior derivative
- $\Delta = \mathrm{d}^* \mathrm{d} = -$ Laplace-Beltrami operator

Δ is essentially s.a.: $\Delta = \int_b^\infty \lambda \, \mathrm{d}P_\lambda, \quad b \geq 0$.

Spectral multipliers: $m : [b, \infty) \to \mathbb{C}$ bounded

$$m(\Delta) = \int_b^\infty m(\lambda) \, \mathrm{d}P_\lambda$$

Riesz transform: $\mathrm{d}\Delta^{-1/2}$

$m(\Delta)$ and $\mathrm{d}\Delta^{-1/2}$ are bounded on $L^2(\mu)$ (easy)

What about $L^p(\mu)$, $p \neq 2$?

Motivation

- Spectral multipliers: fundamental solutions of PDE's

$$\mathrm{e}^{-t\Delta}, \quad \mathrm{e}^{it\Delta}, \quad \cos(t\sqrt{\Delta}), \quad \frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}}$$

- Riesz transform:
  - equivalence of seminorms on $C_c^\infty(M)$

$$\|\mathrm{d}f\|_p \simeq \|\Delta^{1/2}f\|_p?$$

  - regularity of solutions of $\Delta u = f$
  - L^p cohomology of 1 -forms

L^p cohomology

De Rham's theorem (L^2 cohomology): $L^2(\Lambda^1)$ splits orthogonally as

$$L^2(\Lambda^1) = \overline{R(\mathrm{d})} \oplus \overline{R(\mathrm{d}^*)} \oplus \ker(\vec{\Delta})$$

where $\vec{\Delta} = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ is the Hodge-de Rham Laplacian

Is there an analogous decomposition of $L^p(\Lambda^1)$?

Projectors: $\mathrm{d}\Delta^{-1}\mathrm{d}^*, \qquad \mathrm{d}^*\vec{\Delta}^{-1}\mathrm{d}, \qquad \lim_{t\to\infty} e^{-t\vec{\Delta}}.$

Link to Riesz transform

$$\mathrm{d}\Delta^{-1}\mathrm{d}^* = \mathrm{d}\Delta^{-1/2}\left(\mathrm{d}\Delta^{-1/2}\right)^*,$$

$$\mathrm{d}^*\vec{\Delta}^{-1}\mathrm{d} = \mathrm{d}^*\vec{\Delta}^{-1/2}\left(\mathrm{d}^*\vec{\Delta}^{-1/2}\right)^*.$$

Techniques

- extensions of Calderón-Zygmund theory
  [Coulhon-Duong], [Auscher-Coulhon-Duong-Hoffman]
- Littlewood-Paley-Stein functionals **[Lohoué], [Li]**
- Atomic Hardy spaces **[Auscher, MacIntosh, Russ], [Carbonaro-MacIntosh-Morris], [M, Meda, Vallarino], [Carbonaro, M, Meda]**
- Probabilistic methods **[Bakry], [Li]**
- Bellman function **[Carbonaro-Dragicević]**

C-Z theory: basic issues

- The F. T. $\mathcal{F}$ is no longer available to obtain estimates of the kernels.
- In general the C-Z theory does not apply, e.g. if the measure μ is not doubling (manifolds of exponential growth).
- Even when μ is doubling and the kernels can be estimated, they do not satisfy the standard estimates at ∞.

Riesz T. and Heat Kernel

Heat equation $\begin{cases} \partial_t u(x,t) + \Delta u(x,t) = 0 \\ u(x,0) = f(x) \end{cases}$

Solution $u(x,t) = \mathrm{e}^{-t\Delta} f(x) = \int_M h_t(x,y)\, f(y)\, \mathrm{d}\mu(y)$

$h_t(x,y)$ is the heat kernel on M.

Relation with $\mathrm{d}\Delta^{-1/2}$:

$$\mathrm{d}\,\Delta^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, \mathrm{d}\,\mathrm{e}^{-tL} \mathrm{d}t$$

$$k_{\mathrm{d}\Delta^{-1/2}}(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, \mathrm{d}_x h_t(x,y)\, \mathrm{d}t$$

Heat kernel estimates

In $\mathbb{R}^n$

$$h_t(x,y) = (4\pi t)^{-n/2}\, \mathrm{e}^{-|x-y|^2/4t}$$

$$= \frac{c_n}{V(x,\sqrt{t})}\, \mathrm{e}^{-\rho(x,y)^2/4t}$$

Here $V(x,\sqrt{t}) = \mu(B(x,\sqrt{t}))$

DEF M is **upper Li – Yau** if $\exists\ c > 0$ such that

$$h_t(x,y) \le \frac{C}{V(x,\sqrt{t})}\, \mathrm{e}^{-c\,\rho(x,y)^2/4t} \qquad \forall x,y \in M,\ t > 0.$$

Riesz T. and Heat Kernel II

Theorem (Coulhon, Duong)

Suppose that

(1) μ *is doubling*

(2) M *is upper Li-Yau.*

Then $d\Delta^{-1/2}$ *is bounded on* $L^p(M)$ *for* $1 < p < 2$ *and is of weak type* $(1,1)$*.*

Remark

- Assumptions (1), (2) are satisfied if $\mathrm{Ric}\,(M) \geq 0$;
- There are manifolds satisfying (1), (2) for which $\mathrm{d}\Delta^{-1/2}$ is unbounded on $L^p(M)$ for all $p > 2$

Riesz T. and Heat Kernel III

Theorem (Auscher, Coulhon, Duong, Hoffman)

Suppose that M is Li-Yau and μ is doubling. Then TFAE for a given $p_0 > 2$

- $d\Delta^{-1/2}$ *is bounded on* $L^p(M)$ *for all* $2 < p < p_0$ *;*
- $$\| d_x h_t(\bullet, y)\|_p \le \frac{C_p}{\sqrt{t}\,\left[V(y,\sqrt{t})\right]^{1-1/p}} \qquad \forall (y,t) \in M \times \mathbb{R}_+$$
  for all $2 < p < p_0$ *.*
- $$\| d\, \mathrm{e}^{-t\Delta}\|_{p\to p} \le \frac{C_p}{\sqrt{t}} \qquad \forall t \in \mathbb{R}_+$$
  for all $2 < p < p_0$ *.*

Non-doubling manifolds

Theorem (Coulhon, Duong)

Suppose that

- $inj(M) > 0$, $Ric(M) > -k^2$,
- $b = \inf \sigma_2(\Delta) > 0$.

Then $d\Delta^{-1/2}$ *is bounded on* $L^p(M)$, $1 < p \leq 2$.

However the weak type $(1,1)$ estimate is not known in this case.

Our contribution: replace the weak type $(1,1)$ estimate by an estimate $\mathrm{d}\Delta^{-1/2} : X_k \to L^1$ on suitable Hardy spaces X_k.

Theorem (M, Meda, Vallarino)

- *if* $\nabla^j R_M$ *are bounded for* $j = 0, \ldots, 2k-2$

then the higher order Riesz Transform $\nabla^{2k}\Delta^{-k} : X_k \to L^1$ *and is bounded on* $L^p(M)$ *for* $1 < p < 2$.