Harmonic Vector Fields

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Harmonic maps

- (M, g), (N, h) oriented Riemannian manifolds,

$\phi : M \to N$ a C^∞ map,

$\|d\phi\|^2 = \mathrm{trace}_g\,(\phi^* g) : M \to [0, +\infty)$
Hilbert-Schmidt norm of $d\phi$,

$$E_\Omega(\phi) = \frac{1}{2}\int_\Omega \|d\phi\|^2 dv_g, \quad \phi \in C^\infty(M, N),$$

Dirichlet energy functional,

$\Omega \subset\subset M$ relatively compact domain,

dv_g canonical volume form of (M, g) i.e. locally

$dv_g = \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^n \quad \text{on} \quad U$

(U, x^i) local coordinate system on M,

$$G = \det[g_{ij}],\ g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

- $\phi \in C^\infty(M, N)$ is a *harmonic map* if $\frac{d}{dt}\{E_\Omega(\phi_t)\}_{t=0} = 0$ $\forall\, \Omega \subset\subset M$, $\forall\, \{\phi_t\}_{|t|<\epsilon} \subset C^\infty(M, N)$: $\phi_0 = \phi$, $\mathrm{Supp}(V) \subset \Omega$, $V = \left(\frac{\partial \phi_t}{\partial t}\right)_{t=0} \in C^\infty(M, \phi^{-1}T(N))$, $\phi^{-1}T(N) \to M$ pullback bundle.
- From now on M compact and $E = E_M$.

First variation formula:

$$\frac{d}{dt}\{E(\phi_t)\}_{t=0} = -\int_M h^\phi(\tau(\phi),\ V)dv_g$$

$h^{\phi} = \phi^{-1} h$ pullback of h,

$\tau(\phi) = \mathrm{trace}_g \beta(\phi) \in C^{\infty}(M, \phi^{-1} T(N))$ tension field,

$\beta(\phi) = \nabla^{\phi}_X \phi_* Y - \phi_* \nabla_X Y$ second fundamental form of ϕ,

$X, Y \in C^{\infty}(M, T(M))$,

$\nabla^{\phi} = \phi^{-1} \nabla^N \in \mathcal{C}(\phi^{-1} T(N))$ pullback of ∇^N,

∇, ∇^N Levi–Civita connections of (M, g), (N, h).

Hence

$$\phi \in C^{\infty}(M, N) \quad \text{is harmonic} \quad \Longleftrightarrow \tau(\phi) = 0.$$

$\tau(\phi) = 0$ Euler-Lagrange equations. Locally

$$\tau(\phi)^{\alpha} = \Delta \phi^{\alpha} + g^{ij} \left(\Gamma^{\alpha}_{\beta\gamma} \circ \phi \right) \frac{\partial \phi^{\beta}}{\partial x^i} \frac{\partial \phi^{\gamma}}{\partial x^j}$$

(U, x^i) local coordinate system on M, $\left[g^{ij}\right] = [g_{ij}]^{-1}$,

(V, y^α) local coordinate system on N, $\phi^\alpha = y^\alpha \circ \phi$,

$\Gamma^\alpha_{\beta\gamma}$ Christoffel symbols of $h_{\alpha\beta} = h\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right)$,

$\Delta u = -\mathrm{div}\,(\nabla u)$ Laplace-Beltrami operator of (M, g), $u \in C^2(M)$. Locally

$$\Delta u = -\frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G} g^{ij} \frac{\partial u}{\partial x^j}\right) \text{ on } U. \text{ Hence}$$

$u \in C^\infty(M, N)$ is harmonic $\Longleftrightarrow \forall\,(U, x^i),\ (V, y^\alpha)$:

$$\Delta \phi^\alpha + g^{ij}\left(\Gamma^\alpha_{\beta\gamma} \circ \phi\right) \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} = 0, \qquad 1 \leq \alpha \leq \nu, \qquad (1)$$

$\nu = \dim(N)$, the harmonic map system.

Suffices $\phi \in C^2(M, N)$ for all constructions; but then

(1) quasi-linear elliptic & ϕ harmonic $\Longrightarrow \phi \in C^\infty(M, N)$.

- Examples of harmonic maps:
  - geodesics in Riemannian manifolds;
  - minimal isometric immersions (among Riemannian manifolds);
  - Riemannian submersions with minimal fibres;
  - harmonic morphisms i.e. $\phi : M \to N$ continuous map: $\forall\ v \in L^1_{\rm loc}(V)$, $V \subset N$ open, $\Delta^N v = 0$ in $V \Longrightarrow$ $\Longrightarrow \Delta(v \circ \phi) = 0$ in $U = \phi^{-1}(V)$.

[$\forall\ p \in V$, $\exists\ (V, y^\alpha)$ on N such that $\Delta^N y^\alpha = 0$
(local harmonic coordinates)

ϕ harmonic morphism $\Longrightarrow 0 = \Delta(y^\alpha \circ \phi) = \Delta \phi^\alpha \Longrightarrow$
$\Longrightarrow \phi^\alpha \in C^\infty(U) \Longrightarrow \phi \in C^\infty(M, N)$].

Theorem (B. Fuglede & T. Ishihara, 1979)

If $\dim(M) = n \geq \nu$ then any harmonic morphism is a harmonic map.
If $n < \nu$ harmonic morphisms are constant maps.

Proof based on:

Lemma

$\forall\ p \in N$ and $\forall\ (V, y^\alpha)$ local system of normal coordinates with $p \in V$ and $y^\alpha(p) = 0$, and $\forall\ C,\ C_\alpha\,,\ C_{\alpha\beta} \in \mathbb{R}$ with $C_{\alpha\beta} = C_{\beta\alpha}$, $\exists\ v : V \to \mathbb{R}$ such that $\Delta^N v = 0$ in V and

$$v(p) = C, \qquad \frac{\partial v}{\partial y^\alpha}(p) = C_\alpha\,, \qquad \frac{\partial^2 v}{\partial y^\alpha\,\partial y^\beta}(p) = C_{\alpha\beta}\,.$$

Proof of Lemma 2 by Fluglede: based on a version of the implicit function theorem in infinite dimension;

Proof of Lemma 2 by Ishihara: a mess.

Proof still correct (read together with work by Lipman Bers).

P. Baird & J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monographs, New Series, Vol. 29, Clarendon Press, Oxford, 2003.

L. Bers, *Local behavior of solutions of general linear elliptic equations*, Comm. Pure Appl. Math., 8 (1955), 473-496.

B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble), 28(1978), 107-144.

T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ., 19(1979), 215-229.

• Weak harmonic maps

First approach:

whole N covered by one $\psi = (y^1, \cdots, y^\nu) : N \to \mathbb{R}^\nu$

$W^{1,2}(M,N) = \{\phi : M \to N \,|\, \phi^\alpha \in W^{1,2}(M),\ 1 \leq \alpha \leq \nu\}$

[$u \in L^1_{\mathrm{loc}}(M)$ has a *weak gradient* if $\exists\ Y_u \in L^1_{\mathrm{loc}}(M, T(M))$ such that

$$\int_M g(Y_u\,, X) dv_g = - \int_M u \operatorname{div}(X) dv_g\,,$$

$$X \in C_0^\infty(M, T(M))\,.$$

Y_u uniquely determined up to a set of measure zero, denoted by $Y_u = \nabla u$

$\mathcal{D}(\nabla) = W^{1,2}(M)$ space of all $u \in L^2(M)$ having a weak gradient $\nabla u \in L^2(M, T(M))$;

$\nabla : \mathcal{D}(\nabla) \subset L^2(M) \to L^2(M, T(M))$ (densely defined linear operator of Hilbert spaces)

$\nabla^* : \mathcal{D}(\nabla^*) \subset L^2(M, T(M)) \to L^2(M)$ adjoint of ∇ i.e.

$\mathcal{D}(\nabla^*)$ space of all $X \in L^2(M, T(M))$ such that $\exists\, X^* \in L^2(M)$ with

$$\int_M g(\nabla u, X) dv_g = \int_M u X^* dv_g, \quad u \in \mathcal{D}(\nabla),$$

and $\nabla^* X = X^*$.

$C_0^\infty(M, T(M)) \subset \mathcal{D}(\nabla^*)$ and $\nabla^*|_{C_0^\infty(M,T(M))} = -\mathrm{div}.$

$\Delta : \mathcal{D}(\Delta) \subset L^2(M) \to L^2(M)$ Laplacian i.e.

$\mathcal{D}(\Delta) = \{u \in \mathcal{D}(\nabla) : \nabla u \in \mathcal{D}(\nabla^*)\}$ and $\Delta = \nabla^* \circ \nabla]$.

Back to (M, g) compact orientable Riemannian;

$\phi \in W^{1,2}(M, N)$ is a *weak harmonic* map if

$$\int_U \left\{ g^*(\nabla\phi^\alpha\,,\,\nabla\varphi) + g^{ij}(\Gamma^\alpha_{\beta\gamma} \circ \phi)\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\,\varphi \right\} dv_g = 0$$

$\forall\ \varphi \in C_0^\infty(U)$, $\forall\ (U, x^i)$ on M.

Fundamental problem: existence and (partial) regularity of weak harmonic maps.

Dirichlet problem for harmonic map system
S. Hildebrand & H. Kaul & K. Widman, 1977

Theorem

i) *Existence:* N *complete,* $\partial N = \emptyset$, $\nu \geq 2$, $\Omega \subset\subset M$ *domain* $\mathrm{Sect}(N) \leq \kappa^2$, $p \in N$, $\mu < \min\left\{\frac{\pi}{2\kappa}, i(p)\right\}$, $i(p)$ *injectivity radius of* p, $\varphi \in C(\overline{\Omega}, N) \cap W^{1,2}(\Omega, N)$ *with* $\varphi(\overline{\Omega}) \subset B(p, \mu)$. $\implies \exists$ *unique* $\phi \in W^{1,2}(\Omega, N) \cap L^\infty(\Omega, N)$*:*
$\phi(\overline{\Omega}) \subset B(p, \mu)$, $\quad \phi - \varphi \in W_0^{1,2}(\Omega, N)$,
ϕ *minimizes* E_Ω *among all such maps,*
ϕ *is a weak harmonic map.*

ii) *Interior regularity:* $\phi : M \to N$ *bounded weak harmonic,* $\phi(M) \subset B(p, \mu) \implies \phi \in C(M, N)$.

No discussion here of higher interior regularity or boundary regularity.

- F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne*, C. R. Acad. Sci. Paris Sér. I Math. (8)312 (1991), 591-596.
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- S. Hildebrandt, *Harmonic mappings of Riemannian manifolds. Harmonic mappings and minimal immersions*, (Montecatini, 1984), 1-117, Lecture Notes in Math., 1161, Springer, Berlin, 1985.
- S. Hildebrandt & K-O. Widman, *On the Hölder continuity of weak solutions of quasilinear elliptic systems of second order*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (1)4(1977), 145-178.
- S. Hildebrandt & H. Kaul & K-O. Widman, *An existence theorem for harmonic mappings of Riemannian manifolds*, Acta Math., (1-2)138(1977), 1-16.

Harmonic vector fields

R. Moser, *Unique solvability of the Dirichlet problem for weakly harmonic maps*, Manuscripta Math., (3)105(2001), 379-399.

R. Schoen & K. Uhlenbeck, *Regularity of minimizing harmonic maps into the sphere*, Invent. Math., (1)78(1984), 89-100.

- (M, g) compact oriented Riemannian manifold,

$V : M \to T(M)$ tangent vector field,

G Sasaki metric on $T(M)$

$$\forall\, X, Y \in \mathfrak{X}(T(M)) :$$
$$G(X, Y) = g^{\Pi}(LX, LY) + g^{\Pi}(KX, KY),$$

$g^{\Pi} = \Pi^{-1} g$ pullback metric on $\Pi^{-1} T(M) \to M$,
$\Pi : T(M) \to M$ projection,

$L : T(T(M)) \to \Pi^{-1}T(M)$, $L_v X = (v\,,\,(d_v\Pi)X)$,

$K : T(T(M)) \to \Pi^{-1}T(M)$, $K_v = \gamma_v^{-1} \circ Q_v$ Dombrowski map

$X \in T_v(T(M))$, $v \in T(M)$, $Q_v : T_v(T(M)) \to \mathrm{Ker}(d_v\Pi)$ projection,

$T_v(T(M)) = \mathcal{H}_v \oplus \mathrm{Ker}(d_v\Pi)$,

$\mathcal{H}$ horizontal distribution on $T(M)$ associated to the Levi–Civita connection ∇ of (M, g).

• Hence V smooth map of Riemannian manifolds (M, g) and $(T(M), G)$.

Harmonicity?

Theorem (T. Ishihara & O. Nouhaud, 1977)

The following are equivalent

i) $V : (M, g) \to (T(M), G)$ *is a harmonic map.*

ii) V *is an absolute minimum of*

$$E(X) = \frac{1}{2}\int_M \|dX\|^2 dv_g\,, \quad X \in \mathfrak{X}(M).$$

iii) $\nabla V = 0.$

S. Dragomir & D. Perrone, *Harmonic vector fields. Variational principles and differential geometry*, Elsevier Inc., Amsterdam-Boston-Heidelberg-London-New York-Oxford-Paris-San Diego-San Francisco-Singapore-Sydney-Tokyo, 2012.

T. Ishihara, *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ., 13(1979), 23-27.

O. Nouhaud, *Applications harmoniques d'une variété Riemannienne dans son fibré tangent*, C.R. Acad. Sci. Paris, 284(1977), 815-818.

- Yet:

$$E(V) = \frac{n}{2}\,\mathrm{Vol}(M) + B(V), \; B(V) = \frac{1}{2}\int_M \|\nabla V\|^2 dv_g .$$

Total bending functional, or $\mathfrak{biegung}$ $\Longrightarrow$ parallel vector fields are trivially harmonic maps.

Thus

Domain of $E : C^\infty(M, T(M)) \to [0, +\infty)$ too large.

Look for critical points of $E : \mathfrak{X}(M) \to [0, +\infty)$

[variations of $V \in \mathfrak{X}(M)$ are through vector fields $\{V_t\}_{|t|<\epsilon} \subset \mathfrak{X}(M)$ with $V_0 = V$]

Theorem (O. Gil-Medrano, 2001)

$V \in \mathfrak{X}(M)$ critical point of $E : \mathfrak{X}(M) \to \mathbb{R} \Longrightarrow \nabla V = 0$.

O. Gil-Medrano, *Relationship between volume and energy of unit vector fields*, Diff. Geometry Appl., 15(2001), 137-152.

Hence: Domain of $E : \mathfrak{X}(M) \to [0, +\infty)$ still too large.

- $U(M, g)_x = \{v \in T_x(M) : g_x(v, v) = 1\}$, $x \in M$,

$S^{n-1} \to U(M, g) \to M$ tangent sphere bundle

$\mathfrak{X}^1(M) = C^\infty(U(M, g))$ unit vector fields

$X \in \mathfrak{X}^1(M)$ is a *harmonic vector field* if critical point of $E : \mathfrak{X}^1(M) \to [0, +\infty)$ [variations through unit vector fields]

- First variation formula

$$\frac{d}{dt}\{E(X_t)\}_{t=0} = \int_M g(\Delta X, V) dv_g$$

$\{X_t\}_{|t|<\epsilon} \subset \mathfrak{X}^1(M)$, $X_0 = X$,

$$V_x = \frac{d}{dt}\{t \mapsto X_t\}_{t=0} \in T_x(M), \quad g(X, V) = 0$$

$\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ rough Laplacian

$$\Delta X = -\sum_{i=1}^{n} \left\{ \nabla_{E_i} \nabla_{E_i} X - \nabla_{\nabla_{E_i} E_i} X \right\},$$

$\{E_i : 1 \leq i \leq n\}$ local g-orthonormal frame of $T(M)$.

Symbol of rough Laplacian $\sigma_2(\Delta)_\omega : T_x(M) \to T_x(M)$,

$\omega \in T_x^*(M) \setminus \{0\}$, $x \in M$, $\sigma_2(\Delta)_\omega(v) = \|\omega\|^2 v$, $v \in T_x(M)$.

$\sigma_2(\Delta)_\omega$ isomorphism $\Longrightarrow \Delta$ elliptic.

- Euler–Lagrange equations of constrained variational principle

$$\frac{d}{dt} \{E(X_t)\}_{t=0} = 0, \quad g(X, X) = 1,$$

are

$$\Delta X - \|\nabla X\|^2 X = 0.$$

- S. Dragomir & D. Perrone, *On the geometry of tangent hyperquadric bundles: CR and pseudoharmonic vector fields*, Ann. Global Anal. Geom., 30(2006), 211-238.
- G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann., (2)203)(1995), 325-344.
- G. Wiegmink, *Total bending of vector fields on the sphere* S^3, Diff. Geom. Appl., 6(1996), 219-236.
- C.M. Wood, *On the energy of a unit vector field*, Geom. Dedicata, 64(1997), 319-330.
- C.M. Wood, *The energy of Hopf vector fields*, Manuscripta Math., 101(2000), 71-78.

Harmonic vector fields $X : M \to U(M,g)$ aren't harmonic maps $(M,g) \to (U(M,g),G)$ in general:

Theorem (S.D. Han & J.W. Yim, 1998)

Tension field of $X : (M,g) \to (U(M,g),G)$

$$\tau(X) = \left\{ \{\mathrm{trace}_g R\,(\nabla_{\cdot} X\,,\,X)\cdot\}^H - \tan\,(\Delta X)^V \right\} \circ X.$$

Hence

X *harmonic map* $\iff$

i) $\Delta X - \|\nabla X\|^2 X = 0$ *and*

ii) $\mathrm{trace}_g\,\{R(\nabla_{\cdot} X\,,\,X)\cdot\} = 0.$

Harmonic vector fields on Riemannian tori

S.D. Han & J.W. Yim, *Unit vector fields on spheres which are harmonic maps*, Math. Z., 227(1998), 83-92.

- $d_1, d_2 \in \mathbb{R}^2$ linearly independent

$\Gamma = \{m\, d_1 + n\, d_2 \in \mathbb{R}^2 : m, n \in \mathbb{Z}\}$ lattice

$T^2 = \mathbb{R}^2/\Gamma$ torus

$\pi : \mathbb{R}^2 \to T^2$ projection

Assume: T^2 oriented, $\pi : \mathbb{R}^2 \to T^2$ orientation preserving

J almost complex structure on T^2 (induced by fixed orientation)

g Riemannian metric on T^2

$\{S, W\} \subset T(T^2)$ g-orthonormal frame with $W = JS$

$S^1 \to S(T^2, g) \to T^2$ tangent sphere bundle

$\mathcal{E} = \Gamma^\infty(S(T^2, g))$ • $X \in \mathcal{E}$:

$\varphi, \psi \in C^\infty(T^2, \mathbb{R})$ the (S, W)-*coordinates* of X i.e.

$\varphi = g(X, S), \quad \psi = g(X, W),$

$X = \varphi\, S + \psi\, W, \qquad \varphi^2 + \psi^2 = 1.$

$\mathcal{E} \to C^\infty(T^2, S^1)$, $X \mapsto \varphi + \sqrt{-1}\,\psi$ bijection

• $\alpha : \mathbb{R}^2 \to \mathbb{R}$ is an (S, W)-*angle function* for X if

$X \circ \pi = (\cos\alpha)\, S \circ \pi + (\sin\alpha)\, W \circ \pi.$

$\forall\ (m, n) \in \mathbb{Z}^2$ set

$\mathrm{Per}(m, n) = \{\alpha \in C^\infty(\mathbb{R}^2) : \forall\ \xi \in \mathbb{R}^2$

$\alpha(\xi + d_1) - \alpha(\xi) = 2m\pi, \quad \alpha(\xi + d_2) - \alpha(\xi) = 2n\pi\}$

• $\alpha \in \mathrm{Per}(m, n)$ is a (m, n)-*semiperiodic function.*

Also set

$$\mathcal{W} = \bigcup_{(m,n)\in\mathbb{Z}^2} \mathrm{Per}(m,n).$$

Lemma

a) $\forall\ X \in \mathcal{E}$: $\exists$ *an angle function* $\alpha \in \mathcal{W}$.

$X \in \mathcal{E} \Longrightarrow$ *angle functions of* X *differ by integer multiples of* 2π *and lie in but one* $\mathrm{Per}(m,n)$ *for some* $(m,n) \in \mathbb{Z}^2$.

b) *Let* $\alpha \in C^\infty(\mathbb{R}^2)$. *Then* α *is an angle function for some* $X \in \mathcal{E} \Longleftrightarrow \alpha \in \mathcal{W}$.

Define $\mathrm{htp}^{(S,W)} : \mathcal{E} \to \mathbb{Z}^2$ as follows:

Let $X \in \mathcal{E}$ Lemma 7 $\Longrightarrow$

$\exists$ unique $(m,n) \in \mathbb{Z}^2$: $\{$angle functions of $X\} \subset \mathrm{Per}(m,n)$;

Set $\mathrm{htp}^{(S,W)}(X) = (m,n)$.

Lemma

$X, Y \in \mathcal{E}$ homotopic in $\mathcal{E} \iff \mathrm{htp}^{(S,W)}(X) = \mathrm{htp}^{(S,W)}(Y)$. Homotopy classes of elements of $\mathcal{E}$ are thus classified by the elements of $\mathbb{Z}^2$.

- $\mathfrak{X}$ topological space

$\pi^1(\mathfrak{X})$ homotopy classes of maps $f : \mathfrak{X} \to S^1$ (the *Bruschlinsky group*); $X \in \mathcal{E}$: $(m, n) = \mathrm{htp}^{(S,W)}(X) \in \mathbb{Z}^2$

$\alpha \in \mathrm{Per}(m, n)$ angle function for X

Let $e^{i\alpha} : T^2 \to S^1$, $\left(e^{i\alpha}\right)(p) = e^{i\,\alpha(\xi)}$, $\quad \xi \in \pi^{-1}(p)$, $p \in T^2$

$\Longrightarrow \mathcal{E} \to C^\infty(T^2, S^1)$, $\quad X \mapsto e^{i\alpha}$ bijection

$\Longrightarrow \{[X] : X \in \mathcal{E}\} \approx \pi^1(T^2)$ group isomorphism

Hence Lemma 8 is the calculation of the Bruschlinsky group of the torus i.e. $\pi^1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.

- $\mathfrak{biegung}$ in terms of angle functions:

$$g(\nabla S\,,\ W) = a\,g(S,\cdot) + b\,g(W,\cdot),\ a, b \in C^\infty(T^2),$$

$\hat{S},\ \hat{W} \in \mathfrak{X}(\mathbb{R}^2)$: π-related to $S,\ W$

$$\hat{Z} = (a \circ \pi)\,\hat{S} + (b \circ \pi)\,\hat{W} \in \mathfrak{X}(\mathbb{R}^2)$$

$$Q = \{sd_1 + td_2 \in \mathbb{R}^2 : (s,t) \in [0,1]^2\}$$

$$\hat{g} = \pi^* g$$

$$B(X) = \int_{T^2} \|\nabla X\|^2 dv_g = \int_Q \|\hat{\nabla}\alpha + \hat{Z}\|^2_{\hat{g}}\ \pi^* dv_g.$$

May think of $B : \mathcal{W} \to [0, +\infty)$.

$\exists$ Action of $\mathbb{R}$ on $\mathcal{W}$: $\mathbb{R} \times \mathcal{W} \to \mathcal{W}$, $(r, \alpha) \mapsto \alpha + r$.

Action leaves B invariant and preserves $\mathrm{Per}(m,n) \subset \mathcal{W}$, $\forall\ (m,n) \in \mathbb{Z}^2$.

Read on $\mathcal{E} = C^\infty(S(T^2, g))$ action is $\mathrm{SO}(2) \times \mathcal{E} \to \mathcal{E}$

$$\left(\begin{pmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{pmatrix}, X \right) \mapsto (\cos r)\, X + (\sin r)\, JX .$$

Theorem (G. Wiegmink, 1995)

a) *The following statements are equivalent*

i) $X \in \mathrm{Crit}(B)$.

ii) $\forall$ *angle function* α *of* X

$$\hat{\Delta}\alpha - (Sa + Wb) \circ \pi = 0. \tag{2}$$

iii) (S, W)*-Coordinates* (φ, ψ) *of* X *satisfy*

$$\varphi\Delta\psi - \psi\Delta\varphi - Sa - Wb = 0.$$

Theorem

b) *If $X \in \mathrm{Crit}(B)$ then the full orbit of X under any smooth action of a Lie group on $\mathcal{E}$ leaving B invariant consists of critical points. The set of critical points of B intersects each homotopy class $\mathcal{E}^{(S,W)}_{(m,n)} \in \{[X] : X \in \mathcal{E}\} \subset \pi(T^2, S(T^2))$ exactly in one orbit of the $\mathrm{SO}(2)$-action on $\mathcal{E}$. Therefore, up to this action, there is but one critical point in each class $\mathcal{E}^{(S,W)}_{(m,n)}$.*

c) *Let $u \in C^\infty(T^2)$ and let $\tilde{g} = e^{2u}g$ be a metric on T^2 in the conformal class of g. Then a unit vector field $X \in \mathcal{E}$ is a critical point of B if and only if $e^{-u}X$ is a critical point of $\tilde{B}$.*

Theorem

d) *Let h be a flat metric on T^2 and ∇^h its Levi–Civita connection. There is a h-orthonormal frame $\{S_0, W_0\}$ which is parallel with respect to ∇^h. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the Euclidean inner product and norm on $\mathbb{E}^2$; let us set $D = \|d_1\|^2\|d_2\|^2 - \langle d_1, d_2\rangle^2$. The (S_0, W_0)-angle functions $\lambda_{m,n} : \mathbb{R}^2 \to \mathbb{R}$ of the critical points of B on (T^2, h) in the homotopy class $\mathcal{E}^{(S_0,W_0)}_{(m,n)}$ are given by*

$$\lambda_{m,n}(\xi) = \frac{2\pi}{D}\left[\langle d_1, \xi\rangle\left(m\|d_2\|^2 - n\langle d_1, d_2\rangle\right) + \right. \tag{3}$$

$$\left. + \langle d_2, \xi\rangle\left(n\|d_1\|^2 - m\langle d_1, d_2\rangle\right)\right] + s, \quad s \in \mathbb{R},$$

for any $\xi \in \mathbb{R}^2$. Also

$$B(\lambda_{m,n}) = \frac{\pi}{D}\left(m^2\|d_2\|^2 + n^2\|d_1\|^2 - 2mn\langle d_1, d_2\rangle\right).$$

Theorem

e) *For any critical point $X \in \mathcal{E}$ of $\mathcal{B}$ on (T^2, g) and for any $q \in T^2$ there is a conformal coordinate chart $f : U \to \mathbb{E}^2$ (where $U \subseteq T^2$ is an open neighborhood of q) such that*

$$X = (\cos \lambda_{m,n}) \frac{\partial}{\partial f_1} + (\sin \lambda_{m,n}) \frac{\partial}{\partial f_2}$$

in terms of the Gaussian frame field of f with $\lambda_{m,n}$ as in (3) for suitable $(m, n) \in \mathbb{Z}^2$. When $(m, n) = (0, 0)$ the f-coordinates of X are constant.

Stability

- (M, g) compact orientable Riemannian manifold

$X \in \mathfrak{X}^1(M)$ harmonic vector field

$\mathfrak{X} : M \times I_\delta^2 \to S(M, g)$, $I_\delta = (-\delta, \delta)$, $\delta > 0$,

$X_{t,s}(x) = \mathfrak{X}(x, t, s)$, $x \in M$, $t, s \in I_\delta$, $X_{0,0} = X$,

$$V = \left(\frac{\partial X_{t,s}}{\partial t}\right)_{t=s=0}, \quad W = \left(\frac{\partial X_{t,s}}{\partial s}\right)_{t=s=0}$$

$$\frac{\partial^2}{\partial t\,\partial s}\{B(X_{t,s})\}_{t=s=0} = \int_M g(V\,,\,\Delta W - \|\nabla X\|^2 W)\,dv_g$$

second variation formula

Hence: a stability theory for harmonic vector fields.

Theorem (G. Wiegmink, 1995)

(T^2, g) Riemannian torus
$X_0 \in \mathrm{Crit}(B) \subset \mathcal{E} \Longrightarrow \forall$ smooth 1-parameter variation $\{X_t\}_{|t|<\epsilon} \subset \mathcal{E}$ of X

$$\frac{d^2}{dt^2}\{B(X(t))\}_{t=0} \geq 0. \tag{4}$$

Equality in (4) $\Longleftrightarrow$ $\{X_t\}_{|t|<\epsilon}$ is in first order contact at $t = 0$ with a variation $\{Y_t\}_{|t|<\epsilon} \subset \mathcal{E}$ of X_0 such that $Y(t) \in \mathrm{SO}(2) \cdot X_0$ for any $|t| < \epsilon$.

Question: Stability of harmonic vector fields related to spectrum of rough Laplacian?

Analogs of G. Wiegmink's results for harmonic vector fields on Lorentzian surfaces are available:

S. Dragomir & M. Soret, *Harmonic vector fields on compact Lorentz surfaces*, Ricerche mat. DOI 10.1007/s11587-011-0113-1

Problem: Study *biharmonic vector fields* i.e. unit vector fields X which are critical points of the bi-energy functional

$$E_2(X) = \frac{1}{2} \int_\Omega \|\tau(X)\|^2 \; d\, v_g$$

$$X \in \mathfrak{X}^1(M), \quad \Omega \subset\subset M,$$

(variations through unit vector fields).
When $M = T^2$ is a Riemannian torus, does each homotopy class $\mathcal{E}_{(m,n)}^{(S,W)}$ contain a biharmonic representative?

Weakly harmonic vector fields

Weak covariant derivatives

- (M, g) Riemannian manifold

$T^{r,s}(M) \to M$ vector bundle of tangent (r, s)-tensor fields

$\forall\ \varphi \in \Gamma(T^{r,s}(M))$:

$\|\varphi\| = g^*(\varphi, \varphi)^{1/2} : M \to [0, +\infty)$ is well defined.

We set

$$(\varphi, \psi) = \int_M g^*(\varphi, \psi) dv_g$$

$\forall\ \varphi, \psi \in \Gamma(T^{r,s}(M))$ with $g^*(\varphi, \psi) \in L^1(M)$.

$\nabla : C^\infty(T^{r,s}(M)) \to C^\infty(T^{r,s+1}(M))$ covariant derivative

$\nabla^* : C_0^\infty(T^{r,s+1}(M)) \to C_0^\infty(T^{r,s}(M))$ formal adjoint of ∇

i.e.

$$(\nabla^* h\,,\,\varphi) = (h\,,\,\nabla\varphi),$$

$$h \in C_0^\infty(T^{r,s+1}(M)), \qquad \varphi \in C_0^\infty(T^{r,s}(M)).$$

$\forall\ p \geq 1$, $L^p(T^{r,s}(M))$: (r,s)-tensor fields φ with

$$\|\varphi\|_{L^p(T^{r,s}(M))} = \left(\int_M \|\varphi\|^p dv_g\right)^{1/p} < \infty.$$

$\varphi \in L^1_{\mathrm{loc}}(T^{r,s}(M))$ is *weakly differentiable*

if $\exists\ \psi \in L^1_{\mathrm{loc}}(T^{r,s+1}(M))$:

$$(\psi\,,\,h) = (\varphi\,,\,\nabla^* h), \quad h \in C_0^\infty(T^{r,s+1}(M)).$$

ψ uniquely determined up to a set of measure zero;

Notation: $\psi = \nabla\varphi$ *weak covariant derivative* of φ.

Weakly harmonic vector fields

Sobolev spaces of tensor fields

- $\mathcal{H}_g^{0,p}(T^{r,s}(M)) = L^p(T^{r,s}(M))$

$\mathcal{H}_g^{1,p}(T^{r,s-1}(M)) = \{\varphi \in \mathcal{H}_g^{0,p}(T^{r,s-1}(M)) :$

φ weakly differentiable and $\nabla\varphi \in L^p(T^{r,s}(M))\}$.

Recursively $\forall\ k \geq 2$:

$\mathcal{H}_g^{k,p}(T^{r,s-1}(M)) = \{\varphi \in \mathcal{H}_g^{k-1,p}(T^{r,s-1}(M)) :$

$\nabla\varphi \in \mathcal{H}_g^{k-1,p}(T^{r,s}(M))\}$.

$\mathcal{H}_g^{k,p}(T^{r,s}(M))$ Banach space with

$$\|\varphi\|_{\mathcal{H}_g^{k,p}(T^{r,s}(M))} = \left(\sum_{j=0}^{k} \|\nabla^j \varphi\|^p_{L^p(T^{0,j}(M)\otimes T^{r,s}(M))} \right)^{1/p}.$$

- In particular

$$\mathcal{H}^{1,p}_g(T(M)) = \{X \in L^p(T(M)) : \nabla X \in L^p(T^*(M) \otimes T(M))\},$$

$$\|X\|_{\mathcal{H}^{1,p}_g(T(M))} = \left(\|X\|^p_{L^p(T(M))} + \|\nabla X\|^p_{L^p(T^*(M)\otimes T(M))} \right)^{1/p}.$$

Theorem

$1 \le p < \infty \Longrightarrow \mathcal{H}^{1,p}_g(T(M))$ *separable Banach;*

$1 < p < \infty \Longrightarrow$ *reflexive;*

$p = 2 \Longrightarrow \mathcal{H}^{1,2}_g(T(M))$ *separable Hilbert space.*

$X \in \mathcal{H}^{1,2}_g(T(M))$ is a *weak solution* to $\Delta X - \|\nabla X\|^2 X = 0$ if

$$\int_M \{g^*(\nabla X\,,\,\nabla Y) - \|\nabla X\|^2\, g(X,Y)\}\, dv_g = 0$$

$$\forall\; Y \in \mathfrak{X}^\infty_0(M).$$

$X \in \mathcal{H}^{1,2}_g(T(M))$ unit vector field

X *weakly harmonic vector field* if

weak solution to harmonic vector fields system.

Examples

Radial vector fields

$p \in M$, $U \subset M$ normal coordinate neighborhood of p

$\Longrightarrow r = \mathrm{dist}(p\,,\cdot) : U \setminus \{p\} \to \mathbb{R}$ is smooth.

$\dfrac{\partial}{\partial r} \in \mathfrak{X}(U \setminus \{p\})$ radial vector field i.e.

$$g\left(\frac{\partial}{\partial r}\,, X\right) = X(r),\ \forall\, X \in \mathfrak{X}(U \setminus \{p\}).$$

Radial vector field $\dfrac{\partial}{\partial r}$ is

- unit vector field tangent to geodesics issuing at p;
- outward normal of small geodesic sphere $S(p,a)$.

  [Pfaffian system $\left(\mathbb{R}\dfrac{\partial}{\partial r}\right)^{\perp}$ completely integrable and geodesic spheres $S(p,a)$ maximal integral manifolds]
- weakly harmonic vector field on U.

Normal vector fields on principal orbits

Theorem (G. Nunes & J. Ripoll, 2008)

(M, g) compact orientable Riemannnian manifold, $n \geq 3$, $G \subset \mathrm{Isom}(M, g)$ compact Lie group acting on M with cohomogeneity one. Assume either G has no singular orbits or each singular orbit of G has dimension $\leq n-3$.
N unit vector field orthogonal to principal orbits of G $\Longrightarrow$ $N \in \mathcal{H}^{1,2}_g(T(M))$ and critical point of $\mathfrak{biegung}$ $B : \mathcal{H}^{1,2}_g(T(M)) \to \mathbb{R}$.
$M^* \subset M$ union of principal orbits of G; $H : M^* \to \mathbb{R}$, $H(x) =$ mean curvature of orbit $G(x)$ with respect to N
$\Longrightarrow$ $H \in L^2(M)$ and total bending of N is

$$B(N) = -\int_M \mathrm{Ric}(N, N) dv_g + \int_M H^2 dv_g \, .$$

Question

Existence and partial regularity theory
for weakly harmonic vector fields?

Further open problems:

• Given a Riemannian manifold (M, g) and $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$, let $X^\flat \in \Omega^1(M)$ and $\omega^\sharp \in \mathfrak{X}(M)$ be given by

$$g(X, Y) = X^\flat(Y), \quad g(\omega^\sharp, Y) = \omega(Y), \quad \forall \;\; Y \in \mathfrak{X}(M).$$

Study smooth maps $\phi : M \to N$ of Riemannian manifolds such that for any harmonic vector field $Y \in \mathfrak{X}^1(V)$, defined on the open set $V \subset N$, the vector field $\left(\phi^* Y^\flat\right)^\sharp$ is harmonic on $U = \phi^{-1}(V)$.

• Solve the Dirichlet problem for the harmonic vector fields system on a domain $\Omega \subset M$. See also

E. Barletta, *On the Dirichlet problem for the harmonic vector fields equation*, Nonlinear Analysis, 67(2007), 1831-1846.

Book presentation