Polar orbitopes

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Orbitopes

Let K be a compact Lie group and let $K \to \mathrm{GL}(V)$ be a finite-dimensional representation.

An *orbitope* is by definition the convex envelope of an orbit of K in V (Sanyal-Sottile-Sturmfels)

An interesting class of orbitopes is given by the convex envelope of *adjoint orbits*.

G be a real semisimple connected non compact Lie group

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{Cartan decompostion}$$

K is the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$

$$K \longrightarrow Gl(\mathfrak{p}) \;\; k \mapsto Ad(k)_{|_{\mathfrak{p}}}.$$

Given $x \in \mathfrak{p}$, we consider $\mathcal{O} = K \cdot x$ and its convex hull $\widehat{\mathcal{O}}$

Polar Orbitopes

Let K be a compact Lie group and let $K \to \mathrm{O}(V)$ be a finite-dimensional representation.

the K-representation is called *Polar*, if there is a subspace $\mathfrak{a}$ such that:

- $K\mathfrak{a} = V$;
- If $p \in V$, then $\mathfrak{a} \subset (T_p K \cdot p)^{\perp}$.
- Adjoint action is a polar action $\mathfrak{a} \subset \mathfrak{p}$ (maximal abelian subalgebra);
- By a Theorem of Dadok, up to equivalence, a Polar representation is an adjoint action.

Example

$G = \mathrm{SL}(2, \mathbb{R})$.

$K = \mathrm{SO}(2)$

$\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{so}(2) \oplus \mathfrak{sym}_o(2, \mathbb{R})$

$\mathfrak{p}$ is the set of 2×2 symmetric matrices of trace 0.

$\mathfrak{a}$ is the set of diagonal matrices of trace 0

A K-orbit in $\mathfrak{p}$ is a circle or a point.

Then its convex hull is a disk or a point.

Motivation

$X = G/K$ be symmetric space of non compact type.

$$\begin{array}{rcccl} \rho: & G & \to & \mathrm{GL}(V) & \text{complex irreducible representation} \\ & \cup & & \cup & \text{with finite kernel} \\ & K & \to & \mathrm{U}(V) & \end{array}$$

(There exists an Hermitian product $\langle \cdot, \cdot \rangle$ such that $\rho(K) \subset \mathrm{U}(V)$).

Then we have a G-equivariant map

$$\varphi : X \to \mathbb{P}(\mathrm{Herm}(V)), \qquad hK \mapsto [hh^*].$$

Here we identify $h \equiv \rho(h)$. Note that hh^* is the radial part of h.

$X_\rho = \overline{\varphi(X)}$ is called the *Satake-compactification* of G/K.

Upper half-plane

$\mathbb{H}^2 = \{x + iy : y > 0\} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d} \right)$

Therefore $x + iy = \begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} \cdot i$ and so the embedding φ of $\mathbb{H}^2$ is given by

$$\varphi(x+iy) = \begin{pmatrix} y + \frac{x^2}{y} & \frac{x}{y} \\ \frac{x}{y} & \frac{1}{y} \end{pmatrix} = \begin{pmatrix} y^2 + x^2 & x \\ x & 1 \end{pmatrix}$$

$\mathrm{SL}(2,\mathbb{R})$ has 3 orbits in $\mathbb{P}(\mathbb{R}^3) = \{[A] : A = A^t : A \in M_{2\times 2}(\mathbb{R})\}$:

$$\{\det < 0\} \qquad \underbrace{\{\det = 0\} \qquad \{\det > 0\}}_{\overline{X_\rho}}.$$

The *Satake-compactification* of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is $\mathbb{H}^2 \sqcup \mathbb{P}(\mathbb{R}^2)$.

Bourguingnon-Li-Yau-map

$G = K^{\mathbb{C}}$

$\rho : K^{\mathbb{C}} \to \mathrm{GL}(V)$ be an holomorphic irreducible representation

There exists a unique closed $K^{\mathbb{C}}$-orbit $\mathbb{O}$ in $\mathbb{P}(V)$ (unique complex K-orbit)

K acts Hamiltonian on $\mathbb{O}$ with a momentum map $\Phi : \mathbb{O} \longrightarrow \mathfrak{k}$ which is a diffemorphism onto a coadjoint orbit $\mathcal{O}$.

Example: the momentum map of $SU(2)$ acting on $\mathbb{P}(\mathbb{C}^2)$ is given by

$$\begin{aligned}\Phi([z_1, z_2]) &= \frac{-\mathbf{i}}{2(|z_1|^2 + |z_2|^2)} \begin{pmatrix} \frac{1}{2}(|z_2|^2 - |z_1|^2) & \overline{z_2} z_1 \\ \overline{z_1} z_2 & \frac{1}{2}(|z_1|^2 - |z_2|^2) \end{pmatrix} \\ &= SU(2) \cdot \begin{pmatrix} \frac{\mathbf{i}}{2} & 0 \\ 0 & -\frac{\mathbf{i}}{2} \end{pmatrix}.\end{aligned}$$

Bourguignon-Li-Yau map

Let μ be the unique K-invariant probability measure. The *Bourguignon-Li-Yau map* is given by

$$\Psi : K^{\mathbb{C}}/K \longrightarrow \mathfrak{k} \qquad \Psi(gK) = \int_{\mathbb{O}} \Phi(\sqrt{gg^*} \cdot x)\mathrm{d}\mu(x)$$

Theorem (Biliotti-Ghigi, Amer. J. Math. (2013))

Ψ *extends to the Satake compactification* X_ρ *and* Ψ *is a* K*-equivariant homeomorphism of* X_ρ *onto* $\widehat{\mathcal{O}} = \mathrm{conv}(\mathcal{O})$.

$$X_\rho \longleftrightarrow \widehat{\mathcal{O}}$$

Biliotti and Ghigi studied the *integral coadjoint orbits* in the sense of geometric quantization;

Biliotti, Ghigi and Heinzner studied *coadjoint orbitopes*.

Set up

Let G be a semisimple, connected non compact Lie group.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Pick $x \in \mathfrak{p}$ and $\mathcal{O} = K \cdot x$.

This orbit lies in the sphere and in general in a very complicate way.

We study the face structure of $\widehat{\mathcal{O}}$ in the sense of convex geometry.

Hence the goal is to describe the faces in terms of special submanifolds of $\mathcal{O}$.

Convex Geometry

$E \subset (V, \langle \cdot, \cdot \rangle)$ a compact convex subset.

The *relative interior* of E, denoted $\operatorname{relint} E$, is the interior of E in its affine hull.

A *face* F of E is a convex subset $F \subset E$ with the following property: if $x, y \in E$ and $\operatorname{relint}[x, y] \cap F \neq \emptyset$, then $[x, y] \subset F$. A face is closed

The *extreme points* of E are the points $x \in E$ such that $\{x\}$ is a face.

$x \in E$ is an extreme point of E if and only if x cannot be written in the form $x = \lambda a + (1 - \lambda)b$ with $a, b \in E$ and $\lambda \in (0, 1)$.

$E = \operatorname{conv}(\operatorname{ext} E)$.

If F is a face of a convex set E, then $\operatorname{ext} F = F \cap \operatorname{ext} E$.

Theorem

If E is a compact convex set and F_1, F_2 are distinct faces of E then $\operatorname{relint} F_1 \cap \operatorname{relint} F_2 = \emptyset$. If G is a nonempty convex subset of E which is open in its affine hull, then $G \subset \operatorname{relint} F$ for some face F of E. Therefore E is the disjoint union of its open faces.

Convex Geometry

The *support function* of E is the function

$$h_E : V \to \mathbb{R} \qquad h_E(u) = \max_{x \in E} \langle x, u \rangle. \tag{3}$$

If $u \neq 0$, the hyperplane $H(E, u) := \{x \in E : \langle x, u \rangle = h_E(u)\}$ is called the *supporting hyperplane* of E for u.

$F_u(E) := E \cap H(E, u)$ is a face and it is called the *exposed face* of E defined by u

In general not all faces of a convex subset are exposed

$C_F := \{u \in V : F = F_u(E)\}$ is a *convex cone*. If G is a compact subgroup of $O(V)$ that preserves both E and F, then C_F contains a fixed point of G.

Kostant Polytope

Let $\mathfrak{a}$ be a maximal sublagebra of $\mathfrak{p}$ and let $\mathcal{O} = K \cdot x$ be an adjoint orbit.

We can associated to $\mathcal{O}$ two convex set:

- $\widehat{\mathcal{O}} = \operatorname{conv} \mathcal{O}$
- $\mathcal{O} \cap \mathfrak{a} = \mathcal{W} \cdot x$, where $\mathcal{W} = N_K(\mathfrak{a})/C_K(\mathfrak{a}))$ (Weyl group).
  The convex hull of $P = \operatorname{conv}(\mathcal{O} \cap \mathfrak{a}) = \operatorname{conv}(\mathcal{W} \cdot x)$ is called the Kostant polytope.

Theorem (Kostant)

Let $\pi : \mathfrak{p} \to \mathfrak{a}$ be the orthogonal projection. Then

- *the image of $\pi : \mathcal{O} \to \mathfrak{a}$ is a convex set;*
- $\pi(\mathcal{O}) = \pi(\widehat{\mathcal{O}}) = \operatorname{conv}(\mathcal{O} \cap \mathfrak{a}) = P$ *(polytope);*

Theorem 1

We denote by $\mathcal{F}(\widehat{\mathcal{O}})$ the faces of $\widehat{\mathcal{O}}$ and by $\mathcal{F}(P)$ the faces of P.

K acts on $\mathcal{F}(\widehat{\mathcal{O}})$ and the Weyl group $\mathcal{W}$ acts on $\mathcal{F}(P)$.

Theorem

Let $\sigma \in \mathcal{F}(P)$ and let $\sigma^{\perp}$. Then $K^{\sigma^{\perp}} \cdot \sigma$ is a face of $\mathcal{F}(\widehat{\mathcal{O}})$. Moreover the map $\sigma \mapsto K^{\sigma^{\perp}} \cdot \sigma$ passes to a quotients and the resulting map $\mathcal{F}(P)/W \longrightarrow \mathcal{F}(\widehat{\mathcal{O}})/K$ is a bijection.

Example

$G = SL(2,\mathbb{R})$;

$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2) \oplus \mathfrak{sym}_o(2,\mathbb{R})$

$\mathfrak{a}$ is the set of diagonal matrices of trace 0

A K-orbit in $\mathfrak{p}$ is a circle or a point.

Hence its convex hull is a disk or a point.

The Kostant polytope is a closed interval or a point.

Exposed face

Lemma

$\operatorname{ext}\widehat{\mathcal{O}} = \mathcal{O}$ *and if* F *is a face then* $F \cap \mathcal{O} = \operatorname{ext} F$*.*

We denote by $\mu_{\mathfrak{p}} : \mathcal{O} \hookrightarrow \mathfrak{p}$. Let $\beta \in \mathfrak{p}$.

The function $\mu_{\mathfrak{p}}^{\beta}(x) = \langle x, \beta \rangle$ is the height function corresponding to β.

$F_{\beta}(\widehat{\mathcal{O}}) = \{p \in \widehat{\mathcal{O}} : \max_{\widehat{\mathcal{O}}} \mu_{\mathfrak{p}}^{\beta} = \mu_{\mathfrak{p}}^{\beta}(p)\}$

$\operatorname{ext} F_{\beta}(\widehat{\mathcal{O}}) = \{p \in \mathcal{O} : \max_{\mathcal{O}} \mu_{\mathfrak{p}}^{\beta} = \mu_{\mathfrak{p}}^{\beta}(p)\}$.

$\mu_{\mathfrak{p}}^{\beta}$ is a Morse-Bott function.

Exposed face

Proposition

Let $\mathcal{O} = K \cdot x$ and let $F_\beta(\widehat{\mathcal{O}}) \subset \widehat{\mathcal{O}}$ be an exposed face. Then

- $\operatorname{ext} F_\beta(\widehat{\mathcal{O}})$ *is both a K^β and a $(K^\beta)^0$-orbit (it is connected);*
- $F_\beta(\widehat{\mathcal{O}}) \subset \mathfrak{p}^\beta = \{q \in \mathfrak{p} : [q, \beta] = 0\}$*;*
- $F_\beta(\widehat{\mathcal{O}})$ *is a Polar orbitope with respect to $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$.*

- $T_x\mathcal{O} = [\mathfrak{k}, x]$;
- $(\mathrm{d}\mu_{\mathfrak{p}}^\beta)_x([w,x]) = \langle [w,x], \beta \rangle = \langle w, [x, \beta] \rangle$;
- $\mathrm{Crit}(\mu_{\mathfrak{p}}^\beta) = \mathcal{O} \cap \mathfrak{p}^\beta = (K^\beta)^o \cdot N_K(\mathfrak{a}) \cdot x$;
- $\mathrm{Crit}(\mu_{\mathfrak{p}}^\beta)$ is a finite union of K^β-orbit and $F_\beta(\widehat{\mathcal{O}}) \subset \mathfrak{p}^\beta$;

Critical orbits

Lemma

Then x is a local maximum of $\mu_{\mathfrak{p}}^{\beta}$ if and only if there exists a Weyl chamber $C \subset \mathfrak{a}$ such that $x, \beta \in \overline{C}$.

Proof.

- x is a local maximum point of $\mu_{\mathfrak{p}}^{\beta}$ if and only if the Hessian $D^2\mu_{\mathfrak{p}}^{\beta}(x)$ is negative semidefinite.
- $T_x\mathcal{O} = \bigoplus_{\lambda(x)\neq 0} (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}) \cap \mathfrak{p}$.
- $[x, \xi] = -\sum_{\lambda(x)\neq 0} \lambda(x) z_\lambda$;
- $D^2\mu_{\mathfrak{p}}^{\beta}(x)(w, w) = -\sum_{\lambda(x)\neq 0} \lambda(x)\lambda(\beta)|z_\lambda|^2$.
- x is a local maximum of $\mu_{\mathfrak{p}}^{\beta}$ if and only if $\lambda(x)\lambda(\beta) \geq 0$ for every restricted root λ if and only if there exists a Weyl chamber $\mathcal{C}$ such that $x, \beta \in \overline{\mathcal{C}}$.

□

Critical orbits

Lemma

Let $x' \in \mathcal{W} \cdot x$. Then x' is a local maximum of $\mu_{\mathfrak{p}}^{\beta}$ if and only if $w \in \mathcal{W}$ such that $w \cdot x = x'$ and $w \cdot \beta = \beta$.

Corollary

Let β be a nonzero vector in $\mathfrak{p}$ and let $F_\beta(\widehat{\mathcal{O}})$ be the exposed face of $\widehat{\mathcal{O}}$ defined by β. Then $\operatorname{ext} F_\beta(\widehat{\mathcal{O}})$ is both a K^β and a $(K^\beta)^0$-orbit. Hence a local maximum of the function $\mu_{\mathfrak{p}}^{\beta}$ is a global maximum.

Proof.

- $\operatorname{Crit}(\mu_{\mathfrak{p}}^{\beta}) = \mathcal{O} \cap \mathfrak{p}^\beta = (K^\beta)^o \cdot N_K(\mathfrak{a}) \cdot x$;
- applying the above results we get that $\operatorname{ext} F = \max \mu_{\mathfrak{p}}^{\beta}$ is is both a K^β and a $(K^\beta)^0$-orbit.

□

Group theoretical description of the faces

Proposition

Let F be a nonempty face of $\widehat{\mathcal{O}}$. Then there is an abelian subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that F is an orbitope of $(G^{\mathfrak{s}})^0$, i.e. $F \subset \mathfrak{z}_{\mathfrak{p}}(\mathfrak{s})$ and $\operatorname{ext} F$ is an orbit of $(K^{\mathfrak{s}})^0$. If F is proper, then $\mathfrak{s} \neq \{0\}$.

Corollary

If F is a nonempty face of $\widehat{\mathcal{O}}$, then there exists an abelian subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that F is a $G^{\mathfrak{s}} = K^{\mathfrak{s}} \exp(\mathfrak{p}^{\mathfrak{s}})$ Polar orbitope.

Any face is exposed

Let F be a face of $\widehat{\mathcal{O}}$. Choose a subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that F be an orbitope of $(G^{\mathfrak{s}})$. Let $\mathfrak{a}$ be a maximal subalgebra of $\mathfrak{p}$ containing $\mathfrak{s}$. Set $\sigma := \pi(\operatorname{ext} F)$.

Lemma

$\pi(F) = F \cap \mathfrak{a} = \sigma$ is a proper face of P and $F = K^{\sigma^{\perp}} \cdot \sigma$.

Theorem

All proper faces of $\widehat{\mathcal{O}}$ are exposed.

Proof.

$\sigma := F \cap \mathfrak{a} = F \cap P$ is a proper face of P. Then there is a vector $\beta \in \mathfrak{a}$ such that $\sigma = F_{\beta}(P)$. Since $F \cap \mathfrak{a} = F_{\beta}(\widehat{\mathcal{O}}) \cap \mathfrak{a}$ we get $F = F_{\beta}(\widehat{\mathcal{O}})$. □

Proof of Theorem 1

Theorem

The map

$$\Theta : \mathcal{F}(P)/\mathcal{W} \cong \mathcal{F}(\widehat{\mathcal{O}})/K \qquad [\sigma] \mapsto [K^{\sigma^{\perp}} \cdot \sigma]$$

is a bijection.

Proof.

- F is a face, then there exists $\mathfrak{s} \subset \mathfrak{p}$ such that F is a $G^{\mathfrak{s}}$ Polar orbitopes. Since two abelian maximal subalgebra are conjugate, there exists $k \in K$ such that $k\mathfrak{s} \subset \mathfrak{a}$ and so $[F] \in \mathrm{Im}(\Theta)$;
- $\sigma \cong \sigma'$ if and only if $\Theta([\sigma]) = \Theta([\sigma'])$;

□

Faces and Parabolic subgroup

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$;
- $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and $K \cdot x$ is mapped to $K \cdot ix \subset U \cdot ix$;
- $G \subset U^{\mathbb{C}}$, then G acts on $U \cdot ix$ and $G \cdot ix = K \cdot ix$ (Heinzner-Stötzel);
- If $\beta \in \mathfrak{p}$. Then

$$G^{\beta+} := \{g \in G : \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists}\}$$

$$R^{\beta+} := \{g \in G : \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) = e\}$$

- $G^{\beta+} = G^{\beta} \cdot R^{\beta+}$ is a Parabolic subgroup of G.

Faces and Parabolic subgroup

If F is a face, we define

$$H_F := \{g \in K : gF = F\} = \{g \in K : g \cdot \operatorname{ext} F = \operatorname{ext} F\}$$

$$Q_F := \{g \in G : g \cdot \operatorname{ext} F = \operatorname{ext} F\} \qquad C_F := \{\beta \in \mathfrak{p} : F = F_\beta(\widehat{\mathcal{O}})\}.$$

Denote by $C_F^{H_F}$ the vectors of C_F that are fixed by H_F.

Lemma

if $\beta \in C_F^\beta$, then $H_F = K^\beta$ and F is $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$ Polar orbitope. Moreover, $Q_F = G^{\beta+}$ and $\operatorname{ext} F = G^{\beta+} \cdot x$

Theorem

The set $\{\operatorname{ext} F : F$ a nonempty face of $\widehat{\mathcal{O}}\}$ coincides with the set of all closed orbits of parabolic subgroups of G. Any parabolic subgroup $Q \subset G$ has a unique closed orbit, which equals the set of extreme points of a unique face of $F \subset \widehat{\mathcal{O}}$. If $Q = G^{\beta+}$, then $F = F_\beta(\widehat{\mathcal{O}})$.

Other results

- Let $F \in \mathcal{F}(\widehat{\mathcal{O}})$, we define $S = K \cdot \operatorname{relint} F$. We call S the *stratum* corresponding to the face F.

  The strata give a partition of $\partial\widehat{\mathcal{O}}$. They are smooth embedded submanifolds of $\mathfrak{p}$ and are locally closed in $\partial\widehat{\mathcal{O}}$. For any stratum S the boundary $\overline{S} - S$ is the disjoint union of strata of lower dimension.

- we give a description of the faces of the Polar orbitope in terms of root data. We use the formalism of x-connected subset of simple roots developed by Satake. This gives a combinatorial way to generate all the faces of the orbitope.

What's next?

- Define a BLY map on a general symmetric space of non compact type;
- convexity properties of the gradient momentum map;
- Study the convex envelope of an elliptic orbit;
- Study the BLY for a general Kähler manifold.