An introduction to quantum hyperbolic geometry

From joint works with S. Baseilhac - Workshop-PRIN, Pisa - Febbraio 2013

Two approaches to 3D hyperbolic geometry as a classical field theory

W compact closed oriented 3-manifold.

(1) The fields are the *Riemannian metrics* g on W; the hyperbolic structures on W (if any) are the solutions g_h of the *3D Einstein equation* for the Riemannian signature and with (normalized) negative cosmological constant Λ.

(2)(**Gauge theory)**

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•We consider the *Chern-Simons action*.

•The critical points of the action are the *flat connections* (up to gauge equivalence); equivalently the $PSL(2,\mathbb{C})$-charcters of W (i.e. of $\pi_1(W)$) (admitting a lifting to $SL(2,\mathbb{C})$).

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$$\mathrm{Vol}(\rho) := -\pi^2 \mathrm{Im}(CS(\rho)) \geq 0$$

and the *holonomy* character ρ_h of a hyperbolic structure on W (if any) maximises $\mathrm{Vol}(\rho)$.

Relation between the two field theories

For every Riemannian metric g on W we can define:

- $\mathrm{Vol}(W, g)$;
- The Chern-Simons number $CS(W, g) \in \mathbb{R}/\mathbb{Z}$, via gauge theory with group $SO(3, \mathbb{R})$ over the tangent bundle of W.

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We have:

$$\mathrm{Vol}(\rho_h) = \mathrm{Vol}(W, g_h)$$

$$CS(\rho_h) = CS(W, g_h) - \frac{i}{\pi^2}\mathrm{Vol}(W, g_h)\ .$$

Simplicial formulas problem

Given:

- (W, ρ), ρ being any $PSL(2, \mathbb{C})$-charcter of W;
- (T, b) any *simplicial complex* over W, which carries a simplicial *fundamental class* of W:

$$[W] = \sum_{(\Delta,b)\in(T,b)} *_b(\Delta, b), \quad *_b = \pm 1 \ .$$

[Here T refers to a "naked" triangulation of W by oriented tetrahedra, while b refers to the additional combinatorial structure that converts every tetrahedron into a 3-symplex.]

Determine:

1. A suitable enhancement $\mathcal{T} = (T, b, d)$ of (T, b) so that d encodes ρ.
2. A 3-cochain $c(d) \in C^3(T, b; \mathbb{C}/\mathbb{Z})$

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So that

$$S(W, \rho) := c(d)([W]) \in \mathbb{C}/\mathbb{Z}$$

is a *well defined invariant*, and

$$S(W, \rho_h) = CS(\rho_h) \ .$$

(Long story: Bloch, Dupont, Sah, Neumann, Yang, Zagier ...)

ρ-Encoding via parallel transport along the edges

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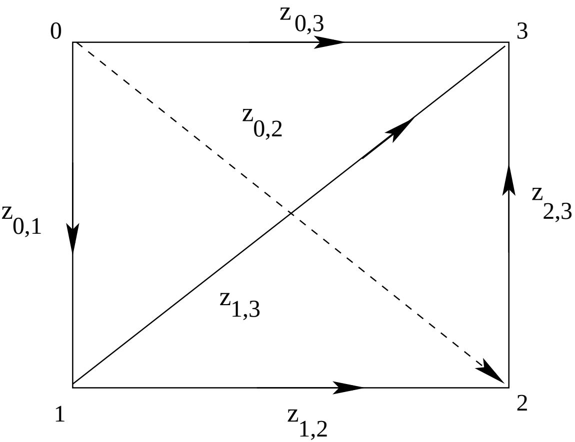
Every b-oriented edge e of (T,b) is labelled by $z(e) \in PSL(2,\mathbb{C})$ so that they verify the four 2-facet relations

$$z_{i,j}z_{j,k}z_{k,i}^{-1},\ \ i<j<k$$

on every 3-symplex (Δ, b).
[Note: This defines also a simplicial 3-cycle of $BPSL(2,\mathbb{C})^{\delta}$.]



$$z_{0,1}\, z_{1,3}\, z_{0,3}^{-1} = 1$$

Idealization: ρ-encoding via cross-ratio systems

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If a 1-cocycle z representing ρ is "generic", then for every 3-symplex the points of S^2_∞ :

$$(p_0, p_1 = z_{0,1}(p_0), p_2 = z_{0,1}z_{1,2}(p_0), p_3 = z_{0,3}(p_0))$$

are distinct and span an hyperbolic ideal tetrahedron. Up to orientation preserving isometries of $\mathbb{H}^3$, this is encoded by a system of *cross-ratios* $\{u_{[i,j]}\}$ which label the edges of (Δ, b).

We stipulate that all edge decorations of a tetrahedron verify the property that opposite edges share the same decoration instance.

Any cross-ratio system on (Δ, b):

$$u_{[i,j]} \in \mathbb{C} \setminus \{0, 1\}$$

is determined by

$$u_0 = u_{[0,1]},\ \ u_1 = u_{[1,2]},\ \ u_2 = u_{[0,2]}$$

and verifies

$$u_{j+1} = 1/(1 - u_j),\ \mathrm{mod}(3)$$

so that

$$\prod_{j=0}^{2} u_j = -1\ .$$

Hence it is eventually determined by u_0. The final cross-ratio enhancement of (Δ, b) takes into account the sign $*_b$:

$$w_j := u_j^{*_b},\ j = 0, 1, 2\ .$$

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•The *gluing variety* $G(T, b)$ is the algebraic subvariety of $(\mathbb{C} \setminus \{0, 1\})^n$ defined by the system of *edge equations*, one for each edge e of T

$$W(e) := \prod_{E \to e} w(E)^{*_b(E)} = 1$$

$W(e)$ is called the *total cross ratio around* e.

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$\bullet$For every symplicial complex (T, b) over W, $G(T, b)$ carries (possibly infinitely many times) all characters of W.

The simplicial “Volume Function”

The *Bloch-Wigner dilogarithm*:

$$\mathrm{D}_2(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x| \ .$$

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We have the *Volume function* defined on the gluing variety:

$$vol : G(T,b) \to \mathbb{R}$$

$$vol(w) := \sum_n (*_b)_n \mathrm{D}_2(w_{0,n})$$

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[Note: This computes the volume of the *scissors coungruence class* of (W, g_h); recall the third Hilbert problem. The *Bloch group*, that organizes these classes, is generated by the ideal tetrahedra.]

Why does $\mathrm{D}_2(x)$ work so well?

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$$\mathrm{D}_2(x^{-1}) = -\mathrm{D}_2(x),\ \mathrm{D}_2(x) = \mathrm{D}_2(1/(1-x))$$

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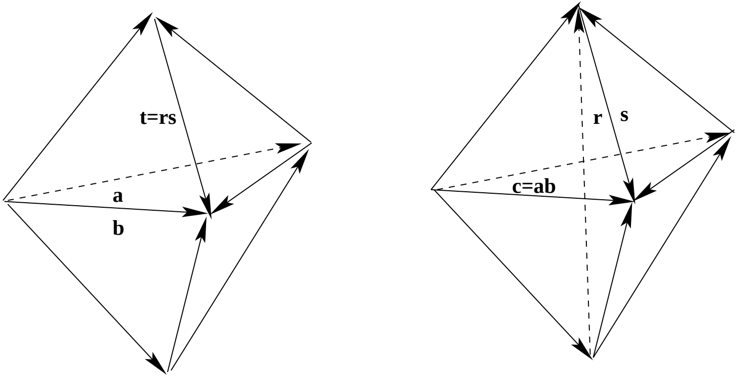
•It verifies all instances of functional *5-terms identities* supported by the enhanced versions

$$(T, b, w) \leftrightarrow (T', b', w')$$

of the basic $2 \leftrightarrow 3$ triangulation move. These define *rational relations* between gluing varieties supported by different simplicial complexes over W.


t=rs
a
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r
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The Rogers dilogarithm

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It is complex analytic on $\mathbb{C} \setminus \{(-\infty; 0) \cup (1; +\infty)\}$ and

$$\mathcal{L}(z) = -\frac{\pi^2}{6} - \frac{1}{2}\log(z)\log(1-z) + \mathrm{Li}_2(z)$$

when $|z| < 1$.

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All the signs $*_b$ *are equal to* 1*, and the cross ratios verify the geometric constraint of representing two subdivisions of a positively oriented ideal convex octahedron.*

We have to perform a clever *analytic continuation* that fixes these defects.

W. Neumann uniformization mod $(\pi^2\mathbb{Z})$

Facts:
Consider the maximal Abelian covering

$$p_0 : \hat{\mathbb{C}} \to \mathbb{C} \setminus \{0, 1\}$$

so that:

•Every point of $\hat{\mathbb{C}}$ can be encoded in the form $[w_0; f_0, f_1, f_2] \in \mathbb{C} \setminus \{0, 1\} \times \mathbb{Z}^2$ so that

$$\sum_{j=0}^{2} \log(w_j) + i\pi f_j = 0 \ .$$

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The integer triple $f := (f_0, f_1, f_0)$ represents a further edge decoration called a *flattening* of (w_0, w_1, w_2), with associated *log-branches*

$$l_j := \log(w_j) + i\pi f_j \ .$$

•The formula

$$\mathrm{L}([w_0; f]) := \mathcal{L}(w_0) + \frac{i\pi}{2}(f_0 \log(1 - w_0) + f_1 \log(w_0))$$

defines an analytic function

$$\mathrm{L} : \hat{\mathbb{C}} \to \mathbb{C}/\pi^2\mathbb{Z}$$

Set

$$p_0 : \hat{\mathbb{C}}^n \to (\mathbb{C} \setminus \{0, 1\})^n \ .$$

The induced infinite covering over the Gluing variety

$\hat{G}(T, b)$ is the complex analytic subset of $p_0^{-1}(G(T, b)) \subset \hat{\mathbb{C}}^n$ defined by the system of *edge equations*, one for each edge e of T

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$$\mathrm{L} : \hat{G}(T,b) \to \mathbb{C}/\pi^2\mathbb{Z}$$

$$\mathrm{L}([w;f]) = \sum_n *_{bn}\mathrm{L}([w_{0,n};f_n]\ .$$

The simplicial "Chern-Simons Function"

Facts:

•Every point $[w; f] \in \hat{G}(T, b)$ represents a couple $(\rho(w), h(f))$ where $h(f) \in H^1(W; \mathbb{Z}/2\mathbb{Z})$.

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•If ρ lifts to $SL(2, \mathbb{C})$ and $h = 0$, then

$$\mathrm{L}(W, \rho) = CS(\rho)$$

this holds in particular for ρ_h.

Tensor networks

If $(\rho, h) = (\rho(w), h(f))$, set:

$$\mathcal{H}(W, \rho, h) := \exp(\frac{2}{i\pi}\mathrm{L}(W, \rho, h)) = \exp(\frac{2}{i\pi}\mathrm{L}([w; f]))$$

and interprete the last term as *the total contraction of a suitable "1-tensor network" carried by* (T, b).

$\bullet(T, b)$ is a network of 3-symplexes connected by a system of oriented arcs transverse to the 2-facets, and contained in the 1-skeleton of the cell decomposition of W dual to (T, b). At each 3-symplex, if $*_b = 1$ the ordered couple of arcs at (F_2, F_0) are outgoing, at (F_3, F_1) they are ingoing. Viceversa if $*_b = -1$.

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•For every $N \geq 1$, (T, b) can be converted into a *N-tensor network* via the following procedure:
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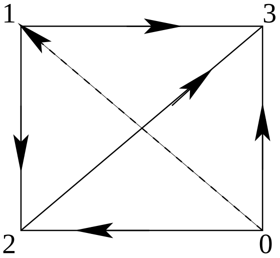
(2) Let every (Δ, b) carry an operator $A = A(\Delta, b) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$, by encoding its matrix elements $A^{p,q}_{r,s}$, $p,q,r,s \in \{0,\dots,N-1\}$, as follows:

$$A = \begin{cases} (A^{i,j}_{k,l}) : V_3 \otimes V_1 \to V_2 \otimes V_0 & \text{if } *_b = 1 \\ (A^{k,l}_{i,j}) : V_2 \otimes V_0 \to V_3 \otimes V_1 & \text{if } *_b = -1 \ . \end{cases} \tag{1}$$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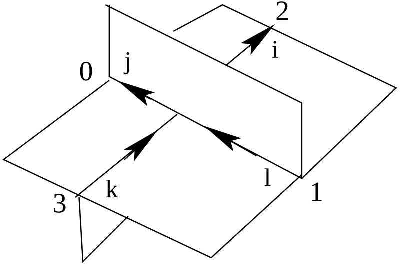
2
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State sum

A *state* σ of a given N-tensor network $\mathcal{A}$ carried by (T, b) is a labelling of the connecting arcs by $\{0, \ldots, N-1\}$. Every state selects a matrix element $A(\Delta, b)_\sigma$ at each 3-symplex. The *state sum*

$$S(\mathcal{A}) := \sum_\sigma \prod_{(\Delta, b)} A(\Delta, b)_\sigma$$

determines a scalar $\in \mathbb{C}$ and is the total contraction of $\mathcal{A}$.

Clearly:
$\bullet\mathcal{H}(W,\rho,h)$ is the total contraction of the 1-tensor network carried by (T,b,w,f), made by the tensors

$$\mathcal{R}(\Delta,b,w,f) := \exp\Big(\frac{2}{i\pi}\mathrm{L}(\Delta,b,w,f)\Big)^{*_b} \ .$$

$\bullet$These tensors verify *multiplicative* 5-terms identities.

Non commutative invariant state sums

Problem:
For $N > 1$, find N-tensor networks $\mathcal{R}_N$ carried by (T, b, w, f), so that the state sums well define invariants $\mathcal{H}_N(W, \rho, h)$.

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One expects both a good behaviour with respect to the tetrahedral symmetries and the verification of the multiplicative and *non-commutative* 5-terms functional identities.

Faddeev-Kashaev $6j$-symbols

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and moreover, $\mathcal{L}_N(\Delta, b, x)$ is a determined *rational function* of x.

[A conceptual explaination is based on De Concini-Kac-Procesi "quantum coadjoint action" theory]

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A tentative solution

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•Let us enhance (T, b) with a further system of edge decoration called *charges*

$$(c_0, c_1, c_2) \in \mathbb{Z}^3$$

one for each (Δ, b), so that

$$c_0 + c_1 + c_2 = 1 \ ,$$

and satisfying the global constraints: for every edge e of T

$$C(e) = \sum_{E \to e} c(E) = 2$$

$C(e)$ is called the *total charge around* e.

•Having fixed a charge c, then for every $[w; f] \in \hat{G}(T, b)$ we have a *system of Nth-roots*

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Then the final N-tensor network $\mathcal{R}_N$ is made by tensors of the form:

$$\mathcal{R}_N(\Delta, b, w, f) = \alpha(w'_N, c)\mathcal{L}_N(\Delta, b, w'_N)$$

where $\alpha(w'_N, c)$ is a "smart" *simmetrizing* scalar factor.

Charge difficulty and actual solution

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Making the theory consistent via a link fixing:
•Let L be a knot or more generally a link in W and assume that (T,H,b) is a *distinguished simplicial complex* over (W,L), so that H is a *Hamiltonian* sub-complex of $T^{(1)}$ that realizes L.

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•Let L be a knot or more generally a link in W and assume that (T, H, b) is a *distinguished simplicial complex* over (W, L), so that H is a *Hamiltonian* sub-complex of $T^{(1)}$ that realizes L.

•A charge c on (T, H, b) is defined as above, provided that for every $e \in H$, the total charge $C(e) = 0$ (instead of $C(e) = 2$). Note that a charge also encodes the link L.

Facts:

(1) Assume that (T, H, b) is as above. Every charge c on (T, H, b) carries a class $k(c) \in H^1(W; \mathbb{Z}/2\mathbb{Z})$. For every $k \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ there exists a charge c such that $k = k(c)$.

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(2) For every couple (W, L) there exist distinguished simplicial complexes (T, H, b).

(3) (like in the classical case) For every couple (ρ, h) there exist $[w; f] \in \hat{G}(T, b)$ so that $(\rho, h) = (\rho(w), h(f))$.

(4) For every (W, L), (T, H, b), $k = k(c)$ and $(\rho, h) = (\rho(w), h(f))$ as above, let $\mathcal{R}_N(T, b, w, f, c)$ be the corresponding N-tensor network.

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Then, up to multiplication by a $2N$-root of unity (*phase anomaly*),

$$\mathcal{H}_N(W, L, \rho, h, k) := S(\mathcal{R}_N(T, b, w, f, c))$$

is a well defined *quantum hyperbolic invariant at level N*.

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(5) Via the above map $w'_N = \exp(\frac{1}{N}(\log(w) + i\pi(N+1)(f - *_b c))$, the state sum function, which is defined on $\hat{G}(T, b)$, is factorized through an algebraic finite covering (of degree N^2)

$$\hat{G}_N(T, b) \to G(T, b)$$

and a determined *rational regular function* defined on $\hat{G}_N(T, b)$.

A special instance: $\mathcal{H}_N(S^3, L)$

The theory is non trivial even at the minimal level of topological complexity:

For every link $L \subset S^3$, up to the phase ambiguity,

$$\mathcal{H}_N(S^3, L) = J_N(L)(\exp(2i\pi/N))$$

where $J_N(L)(q) \in \mathbb{Z}[q^{\pm 1}]$ is the colored Jones polynomial normalized by $J_N(K_U)(q) = 1$ on the unknot K_U.

Some challenging open questions

(1) Improve or even fix the phase anomaly, possibly by introducing further *preserved structures* on the 3-manifolds. At first one expects to improve $2N$ to N, by fixing at least the sign ambiguity.

(2) **QH Asymptotic problem, when $N \to +\infty$.** Let $\mathcal{P}$ be any pattern such the the QHI are defined (this is allusive to the fact that QHI are defined for further patterns such as the the cusped hyperbolic 3-manifolds), then

$$\mathcal{H}_\infty(\mathcal{P}) := \limsup_{N \to +\infty} (\log |\mathcal{H}_N(\mathcal{P})|/N)$$

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is finite.

(a) *Understand when $\mathcal{H}_\infty(\mathcal{P}) = 0$ (sub-exponential case) or when $\mathcal{H}_\infty(\mathcal{P}) \neq 0$ (exponential case).*

(b) *Which "classical" information is carried by $\mathcal{H}_\infty(\mathcal{P})$?*

Volume Conjecture for hyperbolic knots in S^3

A famous particular instance is the Kashaev-Murakami-Murakami volume conjecture for the hyperbolic knots K in S^3:

$$2\pi \lim(\log |J_N(K)(\exp(2i\pi/N)|/N) = Vol(S^3 \setminus K) \ .$$

This is proved for a few knots including the famous figure-8-knot.

(3) Study the relations between different instances of 3-dimensional *quantum invariants* arising from different sectors of the whole representation theory of the quantum group $U_\zeta(sl(2, \mathbb{C}))$.

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