ANALYIS ON PSEUDOCONVEX DOMAINS AND THEIR BOUNDARIES

MARCO M. PELOSO

In this note I intend to present some aspects of analysis and partial differential equations on pseudoconvex domains in $\mathbf{C}^n$ and on related structures. This account should not be considered as the state of the art of research in such a wide field. I will not event attempt to undertake a task of that proportion. I only intent to present my *personal perspective*, based on my personal interests and recent and in-progress works.

Much of what I know is due to my collaborations and thus this presentation should carry my collaborators' names as co-authors.

Contents

1. Introduction

Let Ω be a domain in $\mathbf{C}^n$, $n \geq 1$,

$$\Omega = \big\{ z \in \mathbf{C}^n : \, \varrho(z) < 0 \big\} \, .$$

The function ϱ is called the definining function for Ω and we assume that $\varrho \in C^k(\mathbf{C}^n)$, $k \geq 1$, and that $\nabla\varrho \neq$ on $\{z : \varrho(z) = 0\} = b\Omega$, the boundary of Ω. The domain Ω then is said to have C^k boundary. We have analogous definitions assuming that ϱ is C^∞ or just in the Lipschitz class Λ_α.

Throughout most of this note we consider the case of smooth domains, but in few instances we will study domains with Lipschitz (but piecewise smooth) boundaries.

A C^2 domain is said to be *pseudoconvex* if the Levi form is positive semidefinite on the complex tangent space at every point $\zeta \in b\Omega$, precisely if,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_k \bar{\partial}_z}(\zeta) v_j \bar{v}_k \geq 0 \, ,$$

for $v \in \mathbf{C}^n$ satisfying the condition $\sum_{j=1}^{n} \partial_{z_j}\varrho(\zeta) v_j = 0$. The vectors satisfying the latter condition form a complex vector subspace of $\mathbf{C}T_\zeta(b\Omega)$, the complexified of the tangent space at $\zeta \in \partial\Omega$.

Workshop "Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis", Scuola Normale Superiore, Pisa, February 28- March 3, 2013.

The domain Ω is said to be strongly (or strictly) pseudoconvex if the Levi form is *strictly* positive definite at every point of $\partial\Omega$.

It is immediate to check that the *unit ball* $B_n = \{z : |z|^2 - 1 < 0\}$ and the *Siegel upper half-space* $\mathcal{U}_n = \{z = (z', z_n) : |z'|^2 - \operatorname{Im} z_n < 0\}$ are strongly pseudoconvex domains. In fact they are biholomorphic equivalent, via the Caley transform in n variables. Such a transform extends to a C^∞-diffeomorphism of the sphere $S^{2n-1} = \partial B_n$ and the boundary of $\mathcal{U}_n$. The boundary $b\mathcal{U}_n$ carries a non-coomutative group structure, given by the so-called *Heisenberg group* H_{n-1}.

A domain Ω is said to be *weakly* pseudoconvex if it is pseudoconvex, but its Levi form has vanishing eigenvalues at some points of the boundary. A portion U of $\partial\Omega$ is said to be *Levi flat* if the Levi form vanishes identically on U.

Many aspects of analysis on domains in $\mathbf{C}^n$ concern with some distiguished operators that we now briefly introduce.

The Bergman kernel and projection. Given a domain Ω in $\mathbf{C}^n$ (not necessarly pseudoconvex), we denote by dV the Lebesgue measure in $\mathbf{C}^n$ and by $L^2(\Omega, dV)$, or simply by $L^2(\Omega)$, the space of functions L^2-integrable with respect to dV. Consider the subspace of $L^2(\Omega)$ of holomorphic L^2-integrable functions $A^2(\Omega)$. We call such this space the *Bergman space*. It turns out that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. The orthogonal projection $\mathcal{P}_\Omega$ of $L^2(\Omega)$ onto $A^2(\Omega)$ is called the Bergman *kernel*. This projection is realized as an integral operator

$$\mathcal{P}_\Omega f(z) = \int_\Omega K(z,w) f(w)\, dV(w)\,.$$

The kernel $K(z,w) = K_\Omega(z,w)$ is called the *Bergman kernel*.

One of the main questions, but certainly not the only ones, is to establish under which geometrical condition on a given domain Ω, the Bergman projection is regular on Sobolev and Lebesgue spaces. In connection with the boundary regularity of biholomorphic mappings, via the renowned so-called *Condition R* of Bell. Precisely, a smoothly bounded domain in $\mathbf{C}^n$ satisfies Condition R if the $\mathcal{P}_\Omega$ preserves $C^\infty(\overline{\Omega})$, that is, if, for any given $r > 0$, there exists $M = M(r)$ such that$\mathcal{P}_\Omega : W^{r+M}(\Omega) \to W^r(\Omega)$ is bounded. Incidentally, it is not known whether there exist domains Ω such $\mathcal{P}_\Omega$ *loses* derivatives.

The $\bar\partial$-Neumann problem. In order to study the inhomogeneous Cauchy–Riemann equations

$$\bar\partial u = f \quad \text{on } \Omega$$

for a given pseudoconvex domain Ω one classical approach, that goes back to Hodge in the case of the exterior differentiation, and to Spencer and Kohn in the case of the Cauchy–Riemann operator $\bar\partial$, is the following.

Consider now the L^2-Hilbert space adjoint $\bar\partial^*$ of $\bar\partial$. The idea is to study the boundary value problem

$$\begin{cases} (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)u = f & \text{on } \Omega \\ u,\, \bar\partial u \in \operatorname{dom} \bar\partial^*\,, \end{cases} \tag{1}$$

where f is a given $(0,q)$ form.

The operator $(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$ that arises is a diagonal operator (w.r.t. the natural basis on $(0,q)$-forms) which is in fact the Euclidean Laplacian times the identity matrix.

The conditions that u and $\overline{\partial} u$ lie in the domain of $\overline{\partial}^*$ leads to the $\bar\partial$-*Neumann boundary conditions*. A fairly simple argument shows that these conditions become, resp.,

$$u \lrcorner \partial\varrho = 0, \quad \bar\partial u \lrcorner \partial\varrho = 0 \quad \text{on } b\Omega.$$

(This also explain the terminology *boundary conditions.*) These boundary conditions are not elliptic, and thus the boundary value problem (1) is not elliptic.

Due to classical papers by Hörmander and Kohn, we know that, if Ω is smooth, bounded and pseudoconvex, then (1) is solvable in $L^2_{(0.q)}(\Omega)$, and that the solution operator N_q is bounded in $L^2_{(0.q)}(\Omega)$, for $1 \leq q \leq n-1$. (The cases $q = 0, n$, that are dual to each other, are of different nature, due to the fact that the operator, say when $q = 0$ reduces to $\bar\partial^*\bar\partial$ and its kernel contains all the holomorphic functions.)

It is interesting to observe that the Bergman projection $\mathcal{P}_\Omega$ and N_1 are related by the formula

$$\mathcal{P}_\Omega = I - \bar\partial^* N_1 \bar\partial \,. \tag{2}$$

Similar formulas hold true on $(0, q)$-forms, with $\mathcal{P}_\Omega$ replaced by the orthogonal projection $\mathcal{P}_{\Omega,(0,q)}$ onto the kernel of $\bar\partial$ in $L^2_{(0.q)}(\Omega)$.

In this note we will present some aspects of the research in different settings, on some strictly pseudoconvex domain and boundary, and on some weakly pseudoconvex domain, such as the Diederich–Fornæss *worm* domain $\mathcal{W}_\mu$.

For an introduction to function theory in several complex variables we refer to the monographs [CheS, Kr, R].

2. Analysis on worm domains

The so-called worm domain is the smoothly bounded domain in $\mathbf{C}^2$

Let $\mathcal{W}_\mu$ denote the domain

$$\mathcal{W}_\mu = \Big\{ (z_1, z_2) \in \mathbf{C}^2 : \big| z_1 - e^{i \log |z_2|^2} \big|^2 < 1 - \eta(\log |z_2|^2) \Big\} ,$$

where

(i) $\eta \geq 0$, η is even, η is convex;
(ii) $\eta^{-1}(0) = [-\mu, \mu]$;
(iii) there exists a number $a > 0$ such that $\eta(x) > 1$ if $|x| > a$;
(iv) $\eta'(x) \neq 0$ if $\eta(x) = 1$.

It was introduced by Diederich and Fornæss in [DFo] (see also [KrPe2]).

The properties of the function η guarantee that the domain $\mathcal{W}_\mu$ is smoothly bounded and pseudoconvex. Moreover, its boundary is strongly pseudoconvex except at the boundary points $(0, z_2)$ for $\big|\log |z_2|^2\big| \leq \mu$. These points constitute an annulus in $\partial\mathcal{W}_\mu$. We remark that, since $\partial\mathcal{W}_\mu$ has real-dimension 3, its complex tangent space has complex dimension 1, so $\partial\mathcal{W}_\mu$ is Levi-flat at the points of the annulus.

This domain was introduced in [DFo] to provide an example of a smoothly bounded pseudoconvex domain that does not admit a pseudoconvex neighborhood basis. $\mathcal{W}_\mu$ turns out to be the source of many counterexamples in several complex variables.

Most notably, Barrett proved that the Bergman projection on $\mathcal{W}_\mu$ is not bounded from the Sobolev space W^s into itself if $s \geq \pi/2\mu$ [Ba].

In an on-going project with S. Krantz and C. Stoppato we are trying to compute the Bergman kernel and studying its mapping properties for $\mathcal{W}_\mu$. The complete solution of this problem is still out of reach, but this project is bringing to light a number of intermediate questions that are of their own interest and present unaspected connections with other areas of analysis in one and several variables.

A few years ago, in collaboration with Krantz we studied the Bergman kernel and projection on two model domains that are simpler than $\mathcal{W}_\mu$ but still played an important role in the work of Barrett and before that of Kiselman [Ki]. In [KrPe1] we computed the asymptotic expansion and proved exact range of regularity for the Bergman kernel and projection resp., for the model domains

$$D_\mu = \Big\{(\zeta_1,\zeta_2)\in \mathbf{C}^2 : \mathrm{Re}(\zeta_1 e^{-i\log|\zeta_2|^2})>0\,,\big|\log|\zeta_2|^2\big|<\mu\Big\} \tag{3}$$

and

$$D'_\mu = \Big\{(z_1,z_2)\in \mathbf{C}^2 :\ \big|\operatorname{Im} z_1 - \log|z_2|^2\big| < \frac{\pi}{2},\ \big|\log|z_2|^2\big|<\mu\Big\}\,. \tag{4}$$

We remark that these two domains are biholomorphically equivalent via the mapping

$$\Phi:\ D'_\mu \ni (z_1,z_2)\mapsto (e^{z_1},z_2)\ni D_\mu\,,$$

but the L^p-mapping properties of the two domains are very different. In particular, $\mathcal{P}_{D'_\mu}$ is bounded if and only if $\big|\frac12-\frac1p\big|<\frac{\pi}{2\pi}$.

One key fact in this analysis stems from the following observation. If Ω denotes any of the domains $\mathcal{W}_\mu, D_\mu$ or D'_μ, from the rotational invariance of Ω with respect to the rotations $\rho_\theta(z)=(z_1,e^{i\theta}z_2)$, one can decompose the Bergman space as

$$A^2(\Omega)=\bigoplus_{j\in\mathbb{Z}} A^2_j(\Omega),$$

where $A^2_j(\Omega)$ consists of the functions $f\in A^2(\Omega)$ satisfying $f\circ\rho_\theta\equiv e^{ij\theta}f$. The space $L^2(\Omega)$ admits a similar decomposition into subspaces $L^2_j(\Omega)$, and we can define $W^s_j(\Omega)=L^2_j(\Omega)\cap W^s(\Omega)$.

This decomposition in practice leads to a reduction to problems for weighted Bergman spaces in one dimension, that are easier to handle, thus raising many interesting questions in more familiar and treatable setting. For instance, it becomes of interest to be able to describe functional properties of the Bergman space, and also on some on some wieghted Bergman space, on the *slit disk*

$$S:=\mathbf{D}\setminus\{(-1,0]\}=\big\{z\in\mathbf{C}:|z|<1,\ z\not\in(-1,0]\big\}\,.$$

The results are still incomplete to state precise theorems. However, in this analysis we are also led to consider the *unbounded smooth worm*

$$\mathcal{W}_\infty=\Big\{(z_1,z_2)\in\mathbf{C}\times\mathbf{C}\setminus\{0\}:\ \big|z_1-e^{i\log|z_2|^2}\big|^2<1\Big\}\,.$$

In [KrPS] we are able to compute the expression for the corresponding Bergman kernel, thus showing in particular that the Bergman space for $\mathcal{W}_\infty$ is non-trivial and infinite dimensional. We aspect, as it would be natural to conjecture, that the the Bergman projection is unbounded on $L^p(\mathcal{W}_\infty)$ if $p\neq2$ and on $W^s(\mathcal{W}_\infty)$ if $s>0$.

Many other questions remain open, particularly all analogous questions arising on the boundary of $\mathcal{W}_\mu$, such as the hypoellipticy of the sub-laplacian on $b\mathcal{W}_\mu$, the (ir-)regularity of the Szegö projection, to name the most obvious ones.

Moreover, when trying to investigate the extendebility to the boundary of biholomorphic mappings between pseudoconvex domains, one should consider first the case of biholomorphic self-maps of $\mathcal{W}_\mu$. However, the space $\mathrm{Aut}\,(\mathcal{W}_\mu)$ of automorphisms of $\mathcal{W}_\mu$ is still unknown. It is certainly of great interest to characterize it.

In a separate project in collaboration with D. Barrett and D. Ehsani [BaEP] we analyze a technical aspect in the proof of the Sobolev irregularity of $\mathcal{P}_{\mathcal{W}_\mu}$, namely the presence of pole at the point $\zeta = \frac{\pi}{2\mu} + i\frac{j+1}{2}$ in the Laplace–Fourier transform of the Bergman kernel for $A^2_j(\mathcal{W}_\mu)$. The residue at this pole turns out to be the obstruction to the regularity of the Bergman projection $\mathcal{P}_\mu$.

Our main theorems are grounded on adjustments to remove such poles. We thereby successfully remove the obstruction to regularity of the Bergman projection on the model domains, D_μ, and construct new projections which preserve the spaces $W^s_j(D_\mu)$:

Theorem 2.1. (Barrett, Ehsani, P.) *Let $\mu > 0$, and D_μ be defined as above. For all $j \in \mathbb{Z}$ there exists a bounded linear projection*

$$T_j : L^2(D_\mu) \to B_j(D_\mu)$$

which satisfies

$$T_j : W^s(D_\mu) \to W^s_j(D_\mu)$$

for every $s \geq 0$.

For the same domain D_μ we prove the following result concerning the standard Bergman projection.

Theorem 2.2. (Barrett, Ehsani, P.) *Let $\mu > 0$, and D_μ be defined as above. Then the Bergman projection $\mathcal{P}_\mu : W^s(D_\mu) \to W^s(D_\mu)$ is bounded if and only if $0 \leq s < \pi/(2\mu)$.*

As one might aspect, these theorems lead to results about the regularity of solutions of the inhomogeneous Cauchy–Riemann equations. However, a difficulty that one encounters concerns the density in the subspace of $W^s(D_\mu)$ of $\bar\partial$-closed forms by forms that are $\bar\partial$-closed in a strictly larger $D_{\mu'}$. Discussion of the technical aspect of such result would require introducing more definitions and notation. We will present these results in [BaEP2].

3. Distinguished differential operators

The question of the Sobolev regularity of the Bergman projection is of great interest in several complex variables as a potential tool in order to prove that biholomorphic maps extend smoothly to C^∞-diffeomorphisms of the boundaries.

Formula (2) relates the Bergman projection and the Neumann operator. It is natural to aspect that these two operators enjoy similar regularity properties, although it is far from obvous how to prove such an equivalence and under which hypotheses. Boas and Struabe [BoS1] proved that on a smoothly bounded pseudoconvex domain the regularity of the Neumann operator N_q, $q \geq 1$, implies the regularity of $\mathcal{P}_{q-1}, \mathcal{P}_q$ and $\mathcal{P}_{q+1}$, and conversely.

Before stating some results concerning this equivalence, we introduce a boundary analogue of the $\bar\partial$-complex and the $\bar\partial$-Neumann operator.

Let Ω be a smooth domain in $\mathbf{C}^n$, not necessarily pseudoconvex. The $\bar\partial$-complex in $\mathbf{C}^n$ induces a complex on the boundary $b\Omega$, called the *tangential Cauchy–Riemann* complex and denoted by $\bar\partial_b$. The corresponding equation $\bar\partial_b u = f$ on $b\Omega$ are called the *tangential Cauchy–Riemann* equations. The importance of these equations stem in an old result by Bochner that says, under suitable hypotheses, $\bar\partial_b u = 0$ on $b\Omega$ if and only if u is the restriction to $b\Omega$ of a holomorphic function on Ω.

Thus, in analogy with the $\bar\partial$-Neumann problem, one is led to consider the operator $(\bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b)$, operator that is called the *Kohn Laplacian* of $b\Omega$ and it is usually denoted as $\Box_b$.

The boundary $b\Omega$ is manifold of real dimension $2n-1$ and its tangent space (bundle) admits a complex subspace (sub-bundle) of complex dimension $n-1$, which is the *complex tangent space* that we have already introduced. The complementary direction in the tangent space is called the *complex normal.*

This notion has been generalized in an abstract setting to what are by now classical and called *CR manifolds.*[1]

Then on an abstract CR manifolds $\mathcal{M}$ there exists an operator $\bar\partial_{\mathcal{M}}$ that generalizes the tangential Cauchy-Riemann operator $\bar\partial_b$ on a boundary of a smooth domain in $\mathbf{C}^n$ and that gives rise to a complex; and the operator $(\bar\partial_{\mathcal{M}}\bar\partial^*_{\mathcal{M}}+\bar\partial^*_{\mathcal{M}}\bar\partial_{\mathcal{M}})=\Box_{\mathcal{M}}$, that is still called the Kohn Laplacian.

Under certain assumption on $\mathcal{M}$, that generalizes the notion of pseudoconvexity, it was proved that the Kohn Laplacian $\Box_{\mathcal{M}}$ admits a bounded inverse, the so-called *complex Green operator* (that is the analogue of the Neumann operator) and that we denoted by G_q, where again $(0,q)$ denotes the degree of the form (in the $\bar\partial_{\mathcal{M}}$-complex). The projection of $L^2_{(0,q)}(\mathcal{M})$ onto the kernel of $\bar\partial_{\mathcal{M}}$ in $L^2_{(0,q)}(\mathcal{M})$ is called the Szegö projection, and it is denoted by S_q.

The property that guarantees that $\Box_{\mathcal{M}}$ has closed range and that its inverse is bounded is property of the signature of the Levi form of $\mathcal{M}$ and that counts the number of positive and negative eigenvalues, or the number of pairs of eigenvalues with opposite signs, and it is called the $Y(q)$ condition (depending on the degree of the forms).

Currently, a very active area of research is analysis and geometry of CR manifolds *of higher codimension*, where the notion of codimension is in two different meanings: the pure topological codimension and the dimension of the complement of the complex tangent space in the complexified tangent bundle, quantity that is called the *CR codimension.* If the CR codimension is 1, then the CR manifold is called of *hupersurface type.*

We are ready to present a pair of results recently obtained in collaboration with P. Harrington. and A. Raich [HPR]. The weak $Y(q)$ condition is a generalization of the classical $Y(q)$, and this new condition has been recently introduced by Harrington. and Raich. It is stricly weaker the the $Y(q)$ and of other alternative notions, such as q-concavity.

The question that we address here of two kinds. On one hand we would like to extend the result on the equivalence between the Sobolev regularity of Bergman projection and of the Neumann operator [BoS1] to their boundary analogue, that is, the case of the Szegö projection and the complex Greeen operator. On the other hand we would like to find the minimal conditions under which such equivalence holds, both in the "solid case" and in the "boundary case". These mininal conditions can be stated assuming the least on the regularity of one set of operators w.r.t. the degree of forms.

We remark that the next result is new even in the case of the Szegö projection and the complex Green operator on the boundary of a smooth bounded domain in $\mathbf{C}^n$. The difficulty is proving this result, w.r.t. to case of the Bergman projection and Neumann operator, rests in the lack of Kohn's weight theory on the boundary.

Theorem 3.1. (Harrington, P., Raich) *Let $\mathcal{M}$ be a smooth, compact, embedded, CR manifold of hypersurface type that satisfies weak $Y(q)$ for some $1\le q\le n-2$. Let $s\ge 0$. If G_q is a continuous operator on $W^{s+2}_{(0,q)}(\mathcal{M})$, then there exists a constant C_r so that*

$$\|S_{q-1}\|_{W^r(\mathcal{M})}+\|S_q\|_{W^r(\mathcal{M})}+\|S'_{q+1}\|_{W^r(\mathcal{M})}\le C_r\|G_q\|_{W^r(\mathcal{M})}$$

for $0\le r\le s$.

If S_{q-1}, S_q, and S'_{q+1} are continuous operators on $W^s_{(0,q-1)}(\mathcal{M})$, $W^s_{(0,q)}(\mathcal{M})$, and $W^s_{(0,q+1)}(\mathcal{M})$, respectively, then G_q is a continuous operator on $W^s_{(0,q)}(\mathcal{M})$ and there exists a constant C_s so

[1]Again we refer to S. Dragomir's lecture for a throughful presentation of analysis and geometry of CR manifolds.

that

$$\|G_q\|_{W^s(\mathcal{M})} \leq C_s\big(\|S_{q-1}\|_{W^s(\mathcal{M})} + \|S_q\|_{W^s(\mathcal{M})} + \|S'_{q+1}\|_{W^s(\mathcal{M})}\big).$$

Here $S'_{q+1} = \bar\partial_b G_q \bar\partial_b^*$, the orthogonal projection S'_{q+1} is not the Szegö projection because it annihilates *harmonic forms*, and it coincides with the Szegö projection when $\mathcal{M} = b\Omega$ for a smooth domain Ω.

The next result is similar in spirit and generalizes the result by Boas and Straube by relaxing a number of assumption and proving the result in the context of *Stein manifolds.*

We briefly state the main results in this setting.

Theorem 3.2. (Harrington, P., Raich) *Let M be a Stein manifold and $\Omega \subset M$ a smooth, bounded domain that satisfies weak $Z(q)$ for some $1 \leq q \leq n-1$. Let $s \geq 0$. If N_q is a continuous operator on $W^{s+2}_{(0,q)}(\Omega)$, then there exists a constant C_r so that*

$$\|P_{q-1}\|_{W^r(\Omega)} + \|P_q\|_{W^r(\Omega)} + \|P'_{q+1}\|_{W^r(\Omega)} \leq C_r\|N_q\|_{W^r(\Omega)}$$

for $0 \leq r \leq s$.

If P_{q-1}, P_q, and P'_{q+1} are continuous operators on $W^s_{(0,q-1)}(M)$, $W^s_{(0,q)}(M)$, and $W^s_{(0,q+1)}(M)$, respectively, then N_q is a continuous operator on $W^s_{(0,q)}(M)$ and there exists a constant C_s so that

$$\|N_q\|_{W^s(\Omega)} \leq C_s\big(\|P_{q-1}\|_{W^s(\Omega)} + \|P_q\|_{W^s(\Omega)} + \|P'_{q+1}\|_{W^s(\Omega)}\big).$$

Corollary 3.3. (Harrington, P., Raich) *Let M be a Stein manifold and $\Omega \subset M$ a smooth, bounded domain that satisfies weak $Z(q)$ for some $1 \leq q \leq n-1$. Then N_q is exactly regular if and only if P_{q-1}, P_q, and P'_{q+1} are exactly regular.*

4. Analysis and partial differential equations on the complex sphere

Goal of this section is to describe a detailed analysis of the Kohn Laplacian such as the analysis of the spectral properties, solution of partial differential equations and related function spaces. While it is desirable to be able to conduct such analysis on general pseudoconvex boundaries, at this time we only have enough information in the simplest case: the sphere in $\mathbf{C}^n$. This clearly is a compact, strongly pseudoconvex hypersurface, or in more general terms, CR manifold of hypersurface type.

The results presented in this part were obtained in collaboration with V. Casarino, and are the on-going projects.

In this part we denote by S the unit sphere in $\mathbf{C}^{n+1}$ We consider the Kohn Laplacian

$$\Box_b = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b\,,$$

and the so-called *sublaplacian*

$$\mathcal{L} = \frac{1}{2}\Big(\Box_b + \Box_{\overline{b}}\Big)\,, \tag{5}$$

that is, the real part of the Kohn Laplacian. We will restrict our attention to the operator $\mathcal{L}$ acting on funtions, hence to a scalar differential operator. It is of great interest however to extend the results that we present here, as well as many others, to the case of higher degree forms.

The sublaplacian $\mathcal{L}$ can be written in terms of the coordinates in $\mathbf{C}^n$

$$\mathcal{L} := -\sum_{1\leq j<k\leq n} M_{jk}\overline{M}_{jk} + \overline{M}_{jk}M_{jk}\,, \tag{6}$$

where $M_{jk} := \overline{z}_j \partial_{z_k} - \overline{z}_k \partial_{z_j}$.

It is well known [Ge] that $\mathcal{L}$ had eigenvalues $\lambda_{\ell,\ell'} = 2\ell\ell' + (n-1)(\ell+\ell')$, $\ell,\ell' \geq 0$ and that $L^2(S)$ admits the orthogonal decomposition

$$L^2\left(S^{2n-1}\right) = \bigoplus_{\ell,\ell'=0}^{+\infty} \mathcal{H}^{\ell,\ell'}, \tag{7}$$

where $\mathcal{H}^{\ell,\ell'}$ is the space of complex spherical harmonics of bidegree (ℓ,ℓ') and is the eigenspace relative to the eigenvalue $\lambda_{\ell,\ell'}$.

Denote by $\pi_{\ell,\ell'}$ the spectral projection from $L^2(S^{2n-1})$ onto $\mathcal{H}^{\ell,\ell'}$. This can be written as

$$\pi_{\ell,\ell'} f(z) = \int_{S^{2n+1}} f(w) Z_{\ell,\ell'}(z,w)\, d\sigma(w),$$

where the kernel $Z_{\ell,\ell'}$ is the *zonal harmonic* and can be written in terms of the Jacobi polynomials, see [CaP2] e.g.

Casarino studied the $L^p - L^2$ boundedness for these projections, obtaining the following sharp result.

Theorem 4.1. ([Ca1, Ca2]) *Let $n \geq 2$ and let ℓ,ℓ' be non-negative integers. Then, for $1 \leq p \leq 2$ we have*

$$\|\pi_{\ell,\ell'}\|_{(L^p,L^2)} \leq C\,(2m+n-1)^{\beta(p)}\,(1+M)^{(n-1)\left(\frac{1}{p}-\frac{1}{2}\right)}, \tag{8}$$

where $m = \min(\ell,\ell')$, $M = \max(\ell,\ell')$, and

$$\beta(p) = \begin{cases} (n-1)(\frac{1}{p}-\frac{1}{2}) - \frac{1}{2} & \text{if} \quad 1 \leq p < p_0 \\ -\frac{1}{2}\left(\frac{1}{p}-\frac{1}{2}\right) & \text{if} \quad p_0 \leq p \leq 2, \end{cases}$$

with $p_0 := 2\frac{2n-1}{2n+1}$.

In the recent work in collaboration with V. Casarino we are concern with classical dispersive partial differential equations for the sublaplacian, such as the Schrödinger and, in a future project, the wave equations. The analysis is quite delicate and uses many symmetry properties of the sphere and the explicit expression of many of the integral operators that are involved.

Ideally we would like to "transfer" these results on compact strongly pseudoconvex hypersurfaces, but this project is still out of reach. However, the so-called *Folland–Stein coordinates*, introduced by Folland and Stein in [FoSt] have been succefully used to approximate strongly pseudoconvex hypersurfaces with coordinates of Heisenberg type that are very closely related to the coordinates on the sphere.

In [CaP2] we consider the linear Schrödinger equation

$$\begin{cases} i\partial_t v + \mathcal{L} v = 0 \\ v(0,z) = v_0\,. \end{cases} \tag{9}$$

We prove Strichartz estimates, that are a family of space-time bounds on solutions of (9), and they wprovide a useful tool to control the norm of the solutions and to prove local well-posedness results.

In order to obtain such estimates we bound the $L^p_t L^q_x$ norm of v by means of a suitable mixed Sobolev norm, denoted by $\|v_0\|_{\mathcal{X}^{(r,s)}}$, of the initial datum.

In order to define the spaces $\mathcal{X}^{(r,s)}$ we fix $M \geq 1$ and set

$$\mathcal{V} = \left\{(\ell,\ell') \in \mathbf{N}^2 : \ell/M \leq \ell' \leq M\ell\right\}. \tag{10}$$

We define $\mathcal{X}^{(r,s)}(S^{2n+1})$ to be the space of all functions $u \in L^2(S^{2n+1})$, spectrally decomposed as $u = \sum_{\ell,\ell'=0}^{\infty} h_{\ell,\ell'}$, $h_{\ell,\ell'} \in \mathcal{H}^{\ell,\ell'}$, such that

$$\sum_{\ell/M<\ell'<M\ell} h_{\ell,\ell'} \in W^r(S^{2n+1})\,,$$

while the complementary sums

$$\sum_{\ell'\le \ell/M} h_{\ell,\ell'}\,,\ \sum_{\ell'\ge M\ell} h_{\ell,\ell'} \in W^s(S^{2n+1})\,.$$

Here W^s denotes the non-isotropic Sobolev space of Folland and Stein, see [FoSt].

The Strichartz estimates that we are able to prove for solutions of (9) are expressed in terms of $\mathcal{X}^{(r,s)}$-norms.

Theorem 4.2. (Casarino, P.) *Let S^{2n-1} denote the unit complex sphere in $\mathbf{C}^n$ and let $\mathcal{L}$ be the sublaplacian. Let $Q = 2n$ be the homogeneous dimension, and let $p \ge 2$, $q < +\infty$ satisfy the admissibility condition*

$$\frac{2}{p} + \frac{Q}{q} = \frac{Q}{2}\,.$$

Define

$$s_n := \begin{cases} 2[1-1/n]\,, & \text{if } n > 2 \\ 4/3\,, & \text{if } n = 2\,. \end{cases}$$

Then, if I is any finite time interval and $s \ge s_1$ or $s > s_n$ for $n > 1$, there exists a constant $C = C(s,I) > 0$ such that any solution v of (9) satisfies the estimate

$$\|v\|_{L^p(I,L^q(S^{2n+1}))} \le C\,\|v_0\|_{\mathcal{X}^{(s/p,2/p)}(S^{2n-1})}\,. \tag{11}$$

We will not discuss the proof of this result here, and instead refer to [CaP2]. We only mention that it involves some *spectrally localized dispersive estimates*, that requires quite delicate estimates of an oscillating sum.

The result leads to a local well-possedness result for the non-linear Schrödiger equation, expressed again in terms of the $\mathcal{X}^{(r,s)}$ spaces. This is done in a paper in preparation, by the same authors.

We mention that, even if the sphere and its CR structure are well know and studied, many classical results of harmonic analysis have been proved in the context of the sublaplacian on the sphere only in recent years– we mention in particular the multiplier theorem for the sublaplacian proved by Cowling, Klima and Sikora [CowKS] and the Littlewood–Paley decomposition for the sublaplacian [CaP3] by Casarino and the present author.

References

[Ba] D. Barrett, Behavior of the Bergman projection on the Diederich-Fornæss worm, *Acta Math.* **168** (1992), 1–10.

[BaEP] D. Barrett, D. Ehsani, M. M. Peloso, *Regularity of projection operators attached to worm domains*, preprint 2013.

[BaEP2] ——— in preparation.

[Bel1] S. R. Bell, Biholomorphic mappings and the $\overline{\partial}$-problem, *Ann. Math.* **114** (1981), 103–113.

[Bel2] ———, Local boundary behaviour of proper holomorphic mappings, Proc. Sympos. Pure Math, vol. 41, American Math. Soc., Providence R.I., 1984, 1–7.

[BelLi] S. Bell and E. Ligocka, A simplification and extension of Fefferman's theorem on biholomorphic mappings, *Invent. Math.* **57** (1980), 283–289.

[BerCha] B. Berndtsson, Bo, P. Charpentier, A Sobolev mapping property of the Bergman kernel, *Math. Z.* **235** (2000), 1–10.

[BoS1] ______, Equivalence of regularity for the Bergman projection and the $\overline{\partial}$-Neumann operator, *Manuscripta Math.* **67** (1990), 25–33.

[BoS2] ______, Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in C^n admitting a defining function that is plurisubharmonic on the boundary, *Math. Z.* **206** (1991), 81–88.

[Ca1] V. Casarino, Norms of complex harmonic projection operators, *Canad. J. Math.* **55** (2003), 1134–1154.

[Ca2] ______, Two-parameter estimates for joint spectral projections on complex spheres, *Math. Z.*, **261** (2009), 245–259.

[CaP1] V. Casarino and M. M. Peloso, L^p -summability of Riesz means for the sublaplacian on complex spheres, *J. Lond. Math. Soc.* **83** (2011), 137–152.

[CaP2] ______Strichartz estimates and the nonlinear Schrödinger equation for the sublaplacian on complex spheres, preprint 2012.

[CaP3] ______, Littlewood-Paley decompositions for sublaplacians on complex spheres, *in preparation.*

[CheS] S.-C. Chen and M.-C. Shaw, *Partial Differential Equations in Several Complex Variables*, American Mathematical Society, Providence, RI, 2001.

[Chr] M. Christ, Global C^∞ irregularity of the $\overline{\partial}$-Neumann problem for worm domains, *J. of the Amer. Math. Soc.* **9** (1996), 1171–1185.

[CowKS] M. Cowling, O. Klima, and A. Sikora, Spectral multipliers for the Kohn sublaplacian on the sphere in $\mathbb{C}^n$, *Trans. Amer. Math. Soc.*, **363** (2011), 611-631.

[DFo] K. Diederich and J. E. Fornæss, Pseudoconvex domains: An example with nontrivial Nebenhülle, *Math. Ann.* **225** (1977), 275–292.

[FoKo] G. B. Folland and J. J. Kohn, *The Neumann Problem for the Cauchy-Riemann Complex*, Princeton University Press, Princeton, NJ, 1972.

[FoSt] G. Folland and E. M. Stein, Estimates for the $\overline{\partial}_b$-complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27** (1974), 429–522.

[Ge] D. Geller, The Laplacian and the Kohn Laplacian for the sphere, *J. Differential Geom.* **15** (1980), 417–435.

[HPR] P. S. Harrington and M. M. Peloso and A. S. Raich, Regularity equivalence of the Szegö projection and the complex Green operator, preprint 2013.

[Ki] C. Kiselman, A study of the Bergman projection in certain Hartogs domains, *Proc. Symposia in Pure Math.*, American Math. Society, Providence, RI, 52, III(1991), 219–231.

[Ko] J. J. Kohn, Quantitative estimates for global regularity, *Analysis and geometry in several complex variables* (Katata, 1997), 97–128, Trends Math., Birkhuser Boston, Boston, MA, 1999.

[Kr] S. G. Krantz, *Function Theory of Several Complex Variables*, 2^{nd}

[KrPe1] S. G. Krantz and M. M. Peloso, Bergman kernel and projection on non-smooth worm domains, *Houston Journal of Mathematics* **34** (2008), 873–950.

[KrPe2] ______ *Journal of Geometric Analysis* **18** (2008), 478–510.

[KrPS] S. G. Krantz, M. M. Peloso, and C. Stoppanto, *in preparation.*

[Lig] E. Ligocka, Remarks on the Bergman kernel function of a worm domain, *Studia Mathematica* 130(1998), 109–113.

[R] M. Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, 108. Springer-Verlag, New York, 1986.

[Siu] Y.-T. Siu, Non-Hölder property of Bergman projection of smooth worm domain, *Aspects of Mathematics—Algebra, Geometry, and Several Complex Variables*, N. Mok (ed.), University of Hong Kong, 1996, 264–304.

Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via C. Saldini 50, I-20133 Milano

E-mail address: marco.peloso@unimi.it