The Torelli problem for Logarithmic bundles of hypersurface arrangements in the projective space

Varietà reali e complesse: geometria, topologia e analisi armonica

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Definition

An *effective reduced divisor* $\mathcal{D}$ on the projective complex space $\mathbb{P}^n$ is a family $\mathcal{D} = \{D_1, \dots, D_\ell\}$ of irreducible hypersurfaces of $\mathbb{P}^n$ such that $D_i \neq D_j$. $\mathcal{D}$ is also called *arrangement*. Moreover, such $\mathcal{D}$ has *normal crossings* if it is locally isomorphic to a union of coordinate hyperplanes of $\mathbb{C}^n$.

Definition

Let $\mathcal{D}$ be an arrangement with normal crossings on $\mathbb{P}^n$ and let f be the homogeneous form of degree d defining $\mathcal{D}$. Let consider the vector bundle $T(log\mathcal{D})$ given as the kernel of the map

$$\mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{(\partial_0 f, \dots, \partial_n f)} \mathcal{O}_{\mathbb{P}^n}(d-1).$$

We call *bundle of differential 1-forms on* $\mathbb{P}^n$ *with logarithmic poles on* $\mathcal{D}$, or simply *logarithmic bundle associated to* $\mathcal{D}$

$$\Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) = T(log\mathcal{D})^\vee(-1).$$

Torelli problem for logarithmic bundles

Is the correspondence $\mathcal{D} \longmapsto \Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$ injective? Can we reconstruct $\mathcal{D}$ from $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$?

Arrangements of hyperplanes with normal crossings in $\mathbb{P}^n$

- Dolgachev-Kapranov 1993, [2]: $1 \leq \ell \leq n+1 \Longrightarrow \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) = \mathcal{O}_{\mathbb{P}^n}^{\ell-1} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1-\ell}$
- $\ell = n+2 \Longrightarrow \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) \cong T\mathbb{P}^n(-1)$
- Dolgachev-Kapranov 1993, [2]; Vallès 2000, [3]: $\ell \geq n+3 \Longrightarrow$ we can reconstruct $\mathcal{D}$ unless the D_i's osculate a rational normal curve of degree n in $\mathbb{P}^n$, $\mathcal{C}_n$, in which case $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$ is isomorphic to $E_{\ell-2}(\check{\mathcal{C}}_n)$, the Schwarzenberger bundle of degree $\ell-2$ associated to $\check{\mathcal{C}}_n$

Theorem (A. 2011, [1])

Let $\mathcal{D} = \{C_1, \dots, C_\ell\}$ be an arrangement of ℓ smooth conics in $\mathbb{P}^2$ with normal crossings and let $\mathcal{H} = \{H_1, \dots, H_\ell\}$ be the arrangement of hyperplanes in $\mathbb{P}^5$ that corresponds to $\mathcal{D}$ by means of the quadratic Veronese map. Let assume that $\ell \geq 9$, $\mathcal{H}$ has normal crossings and the H_i's don't osculate a rational normal curve of degree 5 in $\mathbb{P}^5$. Then

$$\mathcal{D} = \{C \subset \mathbb{P}^2 \text{ smooth conic} \mid H^0(C, \Omega^1_{\mathbb{P}^2}(\log \mathcal{D})^\vee_{|C}) \neq \{0\}\}$$

i.e. $\mathcal{D}$ is the set of *unstables* smooth conics of $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$.

So, if $\ell \geq 9$ then the map $\{C_1, \dots, C_\ell\} \longmapsto \Omega^1_{\mathbb{P}^2}(\log\{C_1, \dots, C_\ell\})$ is generically injective.

Proposition (A. 2011, [1])

Let $C \subset \mathbb{P}^2$ be a smooth conic. Then $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D}) \cong T\mathbb{P}^2(-2)$.

Theorem (A. 2012, [1])

Let $\mathcal{D}_1 = \{C_1, C_2\}$ and $\mathcal{D}_2 = \{C'_1, C'_2\}$ be arrangements of smooth conics in $\mathbb{P}^2$ with normal crossings. Then $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D}_1) \cong \Omega^1_{\mathbb{P}^2}(\log \mathcal{D}_2)$ if and only if $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same 4 tangent lines.

Remarks

These results naturally generalize, respectively, to the following cases:

- ▷ arrangements of ℓ smooth hypersurfaces of degree $d \geq 2$ in $\mathbb{P}^n$, $n \geq 2$, with normal crossings
- ▷ one smooth quadric in $\mathbb{P}^n$, $n \geq 3$
- ▷ pairs of smooth quadrics with normal crossings in $\mathbb{P}^n$, $n \geq 3$

References

[1] E. Angelini, PhD thesis, supervised by G. Ottaviani and D. Faenzi (2013)
[2] I. Dolgachev, M. Kapranov, Duke Math. J. 71, n. 3, 633-664 (1993)
[3] J. Vallès, Math. Zeit. 233, 507-514 (2000)