POSTER

PALEY–WIENER THEOREMS FOR THE U(n)–SPHERICAL TRANSFORM ON THE HEISENBERG GROUP

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Introduction. Denote by H_n the Heisenberg group, i.e., the real manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(w,u) = \big(z + w, t + u + \tfrac{1}{2}\operatorname{Im}\langle w|z\rangle\big) \qquad \forall z, w \in \mathbb{C}^n, \quad t, u \in \mathbb{R},$$

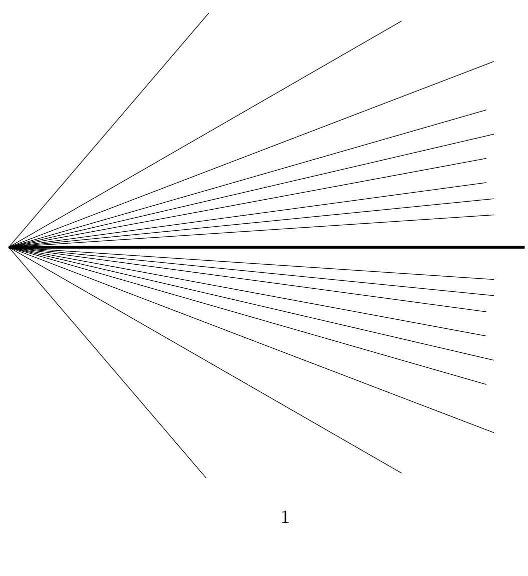
where $\langle\cdot|\cdot\rangle$ denotes the Hermitian scalar product in $\mathbb{C}^n$. The unitary group $\mathrm{U}(n)$ acts on H_n via automorphisms:

$$k \cdot (z,t) = (kz, t) \qquad \forall (z,t) \in H_n, \quad k \in \mathrm{U}(n).$$

A function f on H_n is $\mathrm{U}(n)$–invariant if and only if it depends only on $|z|$ and t. Such functions are usually called *radial*.

The spherical transform $\mathcal{G}$ for the Gelfand pair $\big(H_n \rtimes \mathrm{U}(n), \mathrm{U}(n)\big)$ maps $\mathrm{U}(n)$–invariant functions on the Heisenberg group H_n to functions on the Heisenberg fan

$$\Sigma = \{(\xi,\lambda) : \lambda \neq 0\,,\ \xi = |\lambda|(2j+n)\,,\ j \in \mathbb{N}\} \cup \{(\xi,0) : \xi \geq 0\} \subset \mathbb{R}^2\ .$$



In [4, 5] we have studied the image of the space $\mathcal{S}_{\rm rad}(H_n)$ of radial Schwartz functions, showing that it consists of the restrictions to Σ of Schwartz functions on $\mathbb{R}^2$.

We present here some of the results of [6], where we prove Paley–Wiener type theorems for the spherical transform $\mathcal{G}$ and its inverse. Here we limit ourselves to the results concerning the images, under $\mathcal{G}$ and $\mathcal{G}^{-1}$ respectively, of C^∞ functions with compact support.

Holomorphic extensions. For $(\xi,\lambda)\in\mathbb{C}^2$, the *spherical function* $\Phi_{\xi,\lambda}$ is defined as the unique $\mathrm{U}(n)$-invariant eigenfunction of the sublaplacian L and of the central derivative $i^{-1}\partial_t$ with eigenvalues ξ and λ respectively, which takes the value 1 at the identity element $(0,0)$ of H_n.

The set Σ identifies the spherical functions $\Phi_{\xi,\lambda}$ which are bounded and the spherical transform of an integrable radial function f on H_n is given by

$$\mathcal{G}f(\xi,\lambda)=\int_{H_n} f(z,t)\Phi_{\xi,\lambda}(-z,-t)\,dz\,dt\ ,\qquad (\xi,\lambda)\in\Sigma\ . \tag{1}$$

For functions f with compact support, the right-hand side makes sense also for $\Phi_{\xi,\lambda}$ unbounded, so that $\mathcal{G}f$ extends to all of $\mathbb{C}^2$.

From the explicit expression for the spherical functions

$$\Phi_{\xi,\lambda}(z,t)=\begin{cases} e^{i\lambda t}\,e^{-\lambda|z|^2/4}\,{}_1F_1\left(\frac{n}{2}-\frac{\xi}{2\lambda},n;\frac{\lambda|z|^2}{2}\right) & \lambda\neq 0\ ,\\[2ex] \mathcal{J}_{n-1}(\xi|z|^2/4) & \lambda=0\ ,\end{cases}$$

where ${}_1F_1$ denotes the confluent hypergeometric function and $\mathcal{J}$ the Bessel function, we derive the following consequence.

Lemma 1. *The function* $(x,y,t,\xi,\lambda)\longmapsto\Phi_{\xi,\lambda}(x+iy,t)$ *extends from* $\mathbb{R}^{2n+1}\times\Sigma$ *to an entire function on* $\mathbb{C}^{2n+3}$.

Taking into account (1) and the inversion formula

$$\mathcal{G}^{-1}F(z,t)=\int_\Sigma F(\xi,\lambda)\,\Phi_{\xi,\lambda}(z,t)\,d\mu(\xi,\lambda)\ ,$$

where μ is the Plancherel measure defined by

$$\int_\Sigma\psi\,d\mu=\frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}}\sum_{j=0}^\infty\binom{j+n-1}{j}\psi(|\lambda|(2j+n),\lambda)\,|\lambda|^n\,d\lambda\qquad\forall\psi\in C_c(\Sigma)\ ,$$

we have

Theorem 2. *Let f be an integrable radial function on H_n with compact support. Then $\mathcal{G}f$ extends to an entire function on $\mathbb{C}^2$.*

Let F be a μ-integrable function on Σ with compact support. Then $\mathcal{G}^{-1}F$ extends to an entire function on the complexification of H_n, $H_n^{\mathbb{C}} \cong \mathbb{C}^{2n+1}$.

Characterization of $\mathcal{G}(C^\infty_{c,\mathrm{rad}})$. In the classical Paley-Wiener theorem on $\mathbb{R}^n$, the Fourier transforms of C^∞-functions supported on the ball of radius R are characterized as the entire functions on $\mathbb{C}^n$ whose restriction to $\mathbb{R}^n$ is Schwartz and whose growth in the imaginary directions is controlled by $e^{R|\mathrm{Im}\, z|}$.

It does not look plausible to have a simple "complex variable" description of the entire functions which are in the range of the spherical, or inverse spherical, transform of the space of radial C^∞-functions with compact support (cf. the comments in Fuhr [11], in a context that is closely related to ours).

We obtain instead analogues of the "real variable" version of the Paley-Wiener theorem in $\mathbb{R}^n$, in the spirit of the works of Bang [7] and Tuan [18], later expanded and refined by Andersen and deJeu [1]. Our model statement is the following [1]: a function f on $\mathbb{R}^n$ is the Fourier transform of a C^∞ function supported on the ball of radius R if and only if it is a Schwartz function and, for some $p \in [1, \infty]$,

$$\limsup_{k\to\infty} \|\Delta^k f\|_p^{\frac{1}{k}} \leq R^2 \ . \tag{2}$$

Since the Fourier transform is essentially self-inverse, this theorem works in both directions, whereas in our case we need two separate theorems. Moreover, possible analogues of the above statement rely on the identification, for each direction, of a differential operator on one side and of a corresponding "norm" on the other side.

The following choices are rather natural:

(i) the sublaplacian on H_n and its eigenvalue ξ on Σ (notice that $\xi \cong \big|(\xi, \lambda)\big|$ on Σ);

(ii) the difference/differential operators $M_\pm$ of Benson, Jenkins and Ratcliff [8] on Σ,

$$M_\pm \psi(\xi, \lambda) = \frac{1}{\lambda}\left(\lambda \partial_\lambda + \xi \partial_\xi\right) \psi(\xi, \lambda) - \frac{n\lambda \pm \xi}{2\lambda^2}\big(\psi(\xi \pm 2\lambda, \lambda) - \psi(\xi, \lambda)\big) \ ,$$

and the Korányi norm on H_n,

$$\rho(z, t) = (|z|^4/16 + t^2)^{\frac{1}{4}} \ .$$

Our Paley-Wiener theorem for $\mathcal{G}$ is:

Theorem 3. *Let f be a radial Schwartz function on H_n. The following conditions are equivalent.*

(1) *f has compact support;*

(2) *for every $h \geq 0$ and every p in $[1,\infty]$, $\limsup_{j\to\infty} \| \xi^h M_+^j \mathcal{G}f \|_{L^p(\Sigma)}^{1/j}$ is finite;*

(3) *there exists p in $[1,\infty]$ such that $\liminf_{j\to\infty} \| M_+^j \mathcal{G}f \|_{L^p(\Sigma)}^{1/j}$ is finite.*

Moreover, if any of these conditions is satisfied, then for every $h \geq 0$ and every p in $[1,\infty]$,

$$\lim_{j\to\infty} \| (1+\xi)^h M_+^j \mathcal{G}f \|_{L^p(\Sigma)}^{1/j} = \max_{(z,t)\in \mathrm{supp} f} \rho(z,t)^2 \ .$$

Our Paley-Wiener theorem for $\mathcal{G}^{-1}$ is:

Theorem 4. *Let F be a Schwartz function on $\mathbb{R}^2$. The following conditions are equivalent.*

(1) *$F_{|\Sigma}$ has compact support;*

(2) *for every $h \geq 0$ and every p in $[1,\infty]$, $\limsup_{j\to\infty} \| \rho^h L^j \mathcal{G}^{-1} F_{|\Sigma} \|_{L^p(H_n)}^{1/j}$ is finite;*

(3) *there exists p in $[1,\infty]$ such that $\liminf_{j\to\infty} \| L^j \mathcal{G}^{-1} F_{|\Sigma} \|_{L^p(H_n)}^{1/j}$ is finite.*

Moreover, if any of these conditions is satisfied, then for every $h \geq 0$ and every p in $[1,\infty]$,

$$\lim_{j\to\infty} \| (1+\rho)^h \ L^j \mathcal{G}^{-1} F_{|\Sigma} \|_{L^p(H_n)}^{1/j} = \max_{(\xi,\lambda)\in \Sigma\cap \mathrm{supp} g} \xi \ .$$

Comments. There is a wide literature on Paley–Wiener theorems on the Heisenberg group. The earliest result is due to Ando [2], followed by Thangavelu [14, 15, 16], Arnal and Ludwig [3], Narayanan and Thangavelu [12]. Results are mostly related to the group (operator-valued) Fourier transform and its inverse, but there are also "spectral" Paley–Wiener theorems. This is the case of the already mentioned paper [11], where the condition of compact support on the transform of a given function is expressed, in terms of spectral analysis of the sublaplacian, by the equivalent condition that the function itself belongs to the image of the spectral projection associated to a compact interval in $\mathbb{R}^+$ (see also Strichartz [13], Bray [9], and Dann and Ólafsson [10] in other contexts).

According to Theorem 2 and the extension theorem in [4] for spherical transforms of radial Schwartz functions, it turns out that, for $f \in C_c^\infty(H_n)$ radial, $\mathcal{G}f$ admits two kinds of extensions: to a Schwartz function on $\mathbb{R}^2$ and to an entire function on $\mathbb{C}^2$. In [6] we prove that the two extensions do not necessarily coincide on $\mathbb{R}^2$.

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