POSTER

PALEY–WIENER THEOREMS FOR THE $U(n)$–SPHERICAL TRANSFORM ON THE HEISENBERG GROUP

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Introduction. Denote by $H_n$ the Heisenberg group, i.e., the real manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t)(w, u) = (z + w, t + u + \frac{1}{2} \text{Im} \langle w | z \rangle) \quad \forall z, w \in \mathbb{C}^n, \quad t, u \in \mathbb{R},$$

where $\langle \cdot | \cdot \rangle$ denotes the Hermitian scalar product in $\mathbb{C}^n$. The unitary group $U(n)$ acts on $H_n$ via automorphisms:

$$k \cdot (z, t) = (kz, t) \quad \forall (z, t) \in H_n, \quad k \in U(n).$$

A function $f$ on $H_n$ is $U(n)$-invariant if and only if it depends only on $|z|$ and $t$. Such functions are usually called radial.

The spherical transform $\mathcal{G}$ for the Gelfand pair $(H_n \rtimes U(n), U(n))$ maps $U(n)$-invariant functions on the Heisenberg group $H_n$ to functions on the Heisenberg fan

$$\Sigma = \{(\xi, \lambda) : \lambda \neq 0, \xi = |\lambda|(2j + n), \ j \in \mathbb{N}\} \cup \{\xi, 0) : \xi \geq 0\} \subset \mathbb{R}^2.$$
In [4, 5] we have studied the image of the space $\mathcal{S}_{\text{rad}}(H_n)$ of radial Schwartz functions, showing that it consists of the restrictions to $\Sigma$ of Schwartz functions on $\mathbb{R}^2$.

We present here some of the results of [6], where we prove Paley–Wiener type theorems for the spherical transform $\mathcal{G}$ and its inverse. Here we limit ourselves to the results concerning the images, under $\mathcal{G}$ and $\mathcal{G}^{-1}$ respectively, of $C^\infty$ functions with compact support.

**Holomorphic extensions.** For $(\xi, \lambda) \in \mathbb{C}^2$, the spherical function $\Phi_{\xi, \lambda}$ is defined as the unique $U(n)$-invariant eigenfunction of the sublaplacian $L$ and of the central derivative $i^{-1} \partial_t$ with eigenvalues $\xi$ and $\lambda$ respectively, which takes the value 1 at the identity element $(0, 0)$ of $H_n$.

The set $\Sigma$ identifies the spherical functions $\Phi_{\xi, \lambda}$ which are bounded and the spherical transform of an integrable radial function $f$ on $H_n$ is given by

$$\mathcal{G} f(\xi, \lambda) = \int_{H_n} f(z, t) \Phi_{\xi, \lambda}(-z, -t) \, dz \, dt , \quad (\xi, \lambda) \in \Sigma .$$

For functions $f$ with compact support, the right-hand side makes sense also for $\Phi_{\xi, \lambda}$ unbounded, so that $\mathcal{G} f$ extends to all of $\mathbb{C}^2$.

From the explicit expression for the spherical functions

$$\Phi_{\xi, \lambda}(z, t) = \begin{cases} e^{i\lambda t} e^{-\lambda |z|^2 / 4} \, _1F_1 \left( \frac{n}{2} - \frac{\xi}{2\lambda}, \frac{n\lambda}{2}; \lambda |z|^2 \right) & \lambda \neq 0 , \\ J_{n-1}(\xi |z|^2 / 4) & \lambda = 0 , \end{cases}$$

where $\, _1F_1$ denotes the confluent hypergeometric function and $J$ the Bessel function, we derive the following consequence.

**Lemma 1.** The function $(x, y, t, \xi, \lambda) \mapsto \Phi_{\xi, \lambda}(x + iy, t)$ extends from $\mathbb{R}^{2n+1} \times \Sigma$ to an entire function on $\mathbb{C}^{2n+3}$.

Taking into account (1) and the inversion formula

$$\mathcal{G}^{-1} F(z, t) = \int_{\Sigma} F(\xi, \lambda) \, \Phi_{\xi, \lambda}(z, t) \, d\mu(\xi, \lambda) ,$$

where $\mu$ is the Plancherel measure defined by

$$\int_{\Sigma} \psi \, d\mu = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \sum_{j=0}^{\infty} \left( j + n - 1 \right) \psi(\lambda |(2j + n), \lambda) |\lambda|^n \, d\lambda \quad \forall \psi \in C_0(\Sigma) ,$$

we have
Theorem 2. Let $f$ be an integrable radial function on $H_n$ with compact support. Then $\mathcal{G}f$ extends to an entire function on $\mathbb{C}^2$.

Let $F$ be a $\mu$-integrable function on $\Sigma$ with compact support. Then $\mathcal{G}^{-1}F$ extends to an entire function on the complexification of $H_n$, $H_n^C \cong \mathbb{C}^{2n+1}$.

Characterization of $\mathcal{G}(C^\infty_{c,rad})$. In the classical Paley-Wiener theorem on $\mathbb{R}^n$, the Fourier transforms of $C^\infty$-functions supported on the ball of radius $R$ are characterized as the entire functions on $\mathbb{C}^n$ whose restriction to $\mathbb{R}^n$ is Schwartz and whose growth in the imaginary directions is controlled by $e^{R|\text{Im}z|}$.

It does not look plausible to have a simple “complex variable” description of the entire functions which are in the range of the spherical, or inverse spherical, transform of the space of radial $C^\infty$-functions with compact support (cf. the comments in Fuhr [11], in a context that is closely related to ours).

We obtain instead analogues of the “real variable” version of the Paley-Wiener theorem in $\mathbb{R}^n$, in the spirit of the works of Bang [7] and Tuan [10], later expanded and refined by Andersen and deJeu [1]. Our model statement is the following [1]: a function $f$ on $\mathbb{R}^n$ is the Fourier transform of a $C^\infty$ function supported on the ball of radius $R$ if and only if it is a Schwartz function and, for some $p \in [1, \infty]$,

$$\limsup_{k \to \infty} \|\Delta^k f\|_p^{1/p} \leq R^2.$$  

(2)

Since the Fourier transform is essentially self-inverse, this theorem works in both directions, whereas in our case we need two separate theorems. Moreover, possible analogues of the above statement rely on the identification, for each direction, of a differential operator on one side and of a corresponding “norm” on the other side.

The following choices are rather natural:

(i) the sublaplacian on $H_n$ and its eigenvalue $\xi$ on $\Sigma$ (notice that $\xi \equiv |(\xi, \lambda)|$ on $\Sigma$);

(ii) the difference/differential operators $M_{\pm}$ of Benson, Jenkins and Ratcliff [8] on $\Sigma$,

$$M_{\pm} \psi(\xi, \lambda) = \frac{1}{\lambda} (\lambda \partial_\lambda + \xi \partial_\xi) \psi(\xi, \lambda) - \frac{n \lambda \pm \xi}{2 \lambda^2} (\psi(\xi \pm 2\lambda, \lambda) - \psi(\xi, \lambda)) ,$$

and the Korányi norm on $H_n$,

$$\rho(z, t) = (|z|^4 / 16 + t^2)^{\frac{1}{2}}.$$  

Our Paley-Wiener theorem for $\mathcal{G}$ is:
Theorem 3. Let $f$ be a radial Schwartz function on $H_n$. The following conditions are equivalent.

(1) $f$ has compact support;

(2) for every $h \geq 0$ and every $p$ in $[1, \infty]$, $\limsup_{j \to \infty} \| \xi^h M_j^h \mathcal{G} f \|^ {1/j}_{L^p(\Sigma)}$ is finite;

(3) there exists $p$ in $[1, \infty]$ such that $\liminf_{j \to \infty} \| M_j^p \mathcal{G} f \|^ {1/j}_{L^p(\Sigma)}$ is finite.

Moreover, if any of these conditions is satisfied, then for every $h \geq 0$ and every $p$ in $[1, \infty]$, $\lim_{j \to \infty} \| (1 + \xi^h) M_j^h \mathcal{G} f \|^ {1/j}_{L^p(\Sigma)} = \max_{(z,t) \in \text{supp} f} \rho(z,t)^2$.

Our Paley-Wiener theorem for $\mathcal{G}^{-1}$ is:

Theorem 4. Let $F$ be a Schwartz function on $\mathbb{R}^2$. The following conditions are equivalent.

(1) $F_{|\Sigma}$ has compact support;

(2) for every $h \geq 0$ and every $p$ in $[1, \infty]$, $\limsup_{j \to \infty} \| \rho^h L^j \mathcal{G}^{-1} F_{|\Sigma} \|^ {1/j}_{L^p(H_n)}$ is finite;

(3) there exists $p$ in $[1, \infty]$ such that $\liminf_{j \to \infty} \| L^j \mathcal{G}^{-1} F_{|\Sigma} \|^ {1/j}_{L^p(H_n)}$ is finite.

Moreover, if any of these conditions is satisfied, then for every $h \geq 0$ and every $p$ in $[1, \infty]$, $\lim_{j \to \infty} \| (1 + \rho^h) L^j \mathcal{G}^{-1} F_{|\Sigma} \|^ {1/j}_{L^p(H_n)} = \max_{(\xi,\lambda) \in \Sigma \cap \text{supp}} \xi$.

Comments. There is a wide literature on Paley–Wiener theorems on the Heisenberg group. The earliest result is due to Ando [2], followed by Thangavelu [14, 15, 16], Arnal and Ludwig [3], Narayanan and Thangavelu [12]. Results are mostly related to the group (operator-valued) Fourier transform and its inverse, but there are also “spectral” Paley–Wiener theorems. This is the case of the already mentioned paper [11], where the condition of compact support on the transform of a given function is expressed, in terms of spectral analysis of the sublaplacian, by the equivalent condition that the function itself belongs to the image of the spectral projection associated to a compact interval in $\mathbb{R}^+$ (see also Strichartz [13], Bray [9], and Dann and Ølafsson [10] in other contexts).

According to Theorem 2 and the extension theorem in [4] for spherical transforms of radial Schwartz functions, it turns out that, for $f \in C_c^\infty(H_n)$ radial, $\mathcal{G} f$ admits two kinds of extensions: to a Schwartz function on $\mathbb{R}^2$ and to an entire function on $\mathbb{C}^2$. In [6] we prove that the two extensions do not necessarily coincide on $\mathbb{R}^2$. 
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REFERENCES


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