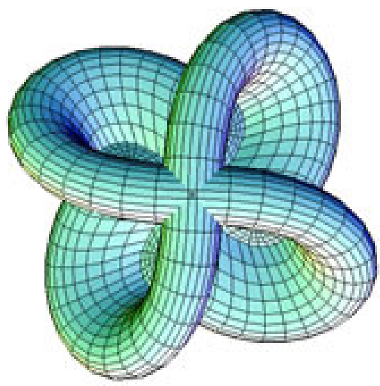

Workshop

varietà reali e complesse:
geometria, topologia e analisi armonica

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Scuola Normale Superiore - Aula Dini

LCK METRICS AND NUMBER FIELDS

Poster by

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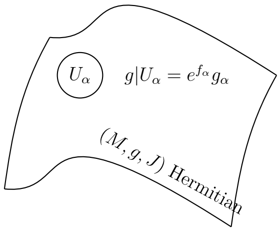
LCK geometry

See PV12 for details

Oeljeklaus-Toma (OT) manifolds are compact complex manifolds associated to number fields, that is, algebraic field extensions of $\mathbb{Q}$. OT manifolds provide a link between number theory and differential geometry, and are a source of many interesting examples. Here we describe in particular their relationship with locally conformally Kähler (LCK) structures.

Definition

Locally conformally Kähler (LCK) manifold: Hermitian manifold (M, g, J) whose metric g is locally conformal to Kähler metrics.



g_α is defined on U_α

Denote by $\mathrm{Hmt}(K)$ the group of biholomorphic homotheties of a Kähler manifold K.

Recipe

Examples of LCK manifolds are given as K/Γ, where $\Gamma \subset \mathrm{Hmt}(K)$ is a discrete Lie group of biholomorphic homotheties acting freely and properly discontinously on K.

Almost every LCK example is built in this way

Alternative definition

A LCK manifold is a pair (K, Γ), where K is a Kähler manifold and $\Gamma \subset \mathrm{Hmt}(K)$ is a discrete Lie group of biholomorphic homotheties acting freely and properly discontinously on K.

$M = K/\Gamma$ as a conformal manifold

Being LCK is a conformal property. This gives the following conformal invariant.

LCK Rank

Denote by $\rho_K : \mathrm{Hmt}(K) \to \mathbb{R}^+$ the homomorphism given by the dilation factors. The rank $\mathrm{rk}(M)$ of the LCK manifold $M = K/\Gamma$ is the rank of the free abelian group $\rho_K(\Gamma)$.

$\mathrm{rk}(\rho_K(\Gamma))$ depends only on the conformal class of K/Γ

OT manifolds

See OT05 for details

Ingredients

- A finite field extension F of $\mathbb{Q}$ of degree $n = s + 2t$.
- The embeddings $\{\sigma_i : F \to \mathbb{C}\}_{i=1,\dots,n}$, where the first s are real.
- The ring $\mathcal{O}_F$ of algebraic integers of F.
- The map $\sigma : F \to \mathbb{C}^{s+t}$ given by $\sigma(x) = (\sigma_1(x), \dots, \sigma_{s+t}(x))$.

OT recipe

Via σ, $\mathcal{O}_F \rtimes \mathcal{O}_F^*$ acts on $\mathbb{C}^{s+t}$. This action is neither free nor properly discontinously. In OT05 it is shown that a certain multiplicative subgroup U of $\mathcal{O}_F^{*,+}$ can be found, such that $\mathcal{O}_F \rtimes U$ acts freely and properly discontinously on $\mathbb{H}^s \times \mathbb{C}^t$, with compact quotient ($\mathbb{H}$ denotes the upper half-plane in $\mathbb{C}$).

Definition

Given a finite field extension F of $\mathbb{Q}$, the compact complex manifold

$$M(F,U) = \frac{\mathbb{H}^s \times \mathbb{C}^t}{\mathcal{O}_F \rtimes U}$$

is called an Oeljeklaus-Toma (OT) manifold.

OT manifolds disproved a long-standing conjecture

LCK geometry and OT manifolds

See OT05, PV12 for details

LCK or not LCK OT05

- For $t = 1$, OT manifolds admit a LCK structure.
- For $s = 1$ and $t > 1$, OT manifolds does not admit any LCK structure.

Open question for $s, t > 1$

Non-trivial rank PV12

For $t = 1$: $\mathrm{rk}(M(F,U)) \neq 1$ or $b_1(M)$ if and only if F is a quadratic extension of a totally real number field.

- Proof relies on the study of units with absolute value 1 in the complex embedding.

First examples of non-trivial LCK rank

No subvarieties OV11

For $t = 1$ the complex manifold $M(F, U)$ has no closed analytic subspaces of proper dimension.

- Proof relies on a deeply technical number-theoretical fact ("adelic strong approximation").

OT manifolds as Inoue surfaces

Not LCK Vul13

LCK structure on $M(F, U)$ for $s, t > 1 \Leftrightarrow$ All units in U have all complex Galois conjugates of equal absolute value. Consequence: no LCK structure if $2s < t$.

- Proof relies on the study of polynomials with many roots of equal absolute value.

Partial answer

Torsion of $H_1(M)$

$[\Gamma, \Gamma]$ is isomorphic to the ideal of $\mathcal{O}_F$ generated by all $1 - u$, with $u \in U$. In particular, if U contains an "exceptional unit", then $H_1(M(F, U))$ is torsion-free.

- Existence of exceptional units in number fields is an important and well-studied topic.

Work in progress

Mahler

If $t = 1$, the volume of the LCK metric carries information about discriminant and regulator of F. Also, the length of closed geodesics is related to the "Mahler measure" of units in U. To be explored.

Work in progress

References

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