Formality and symplectic structures of almost abelian solvmanifolds

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Introduction

A solvmanifold S is a compact homogeneous space $S = G/\Gamma$, where G is a connected and simply connected solvable Lie group and Γ is a lattice in G (i.e. a discrete subgroup with compact quotient space).

S is almost abelian if $G = \mathbb{R} \ltimes_{\varphi} \mathbb{R}^n$ and $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^n$. In particular $\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{ad}_{X_{n+1}}} \mathbb{R}^n$, where $\varphi(t) = exp(t\,\mathrm{ad}_{X_{n+1}})$, then $[X_i, X_j] = 0 \quad \forall\, i, j < n+1$.

$$\bigwedge \mathfrak{g}^* := \{G\text{-invariant forms on } G\} \subset \bigwedge S := \{\Gamma\text{-invariant forms on } G\}.$$

In general $H^*(\mathfrak{g}) \subsetneq H^*(S)$, but when the Mostow condition holds, i.e. for the algebraic closures $\mathcal{A}(\mathrm{Ad}_G(\Gamma)) = \mathcal{A}(\mathrm{Ad}_G(G))$, then $H^*(\mathfrak{g}) = H^*(S)$, [Mostow].

When the Mostow condition does not hold it is quite difficult to compute the cohomology of a solvmanifold. In particular in 1997 Tralle and Oprea developed a method to compute the de Rham cohomology of almost abelian solvmanifolds using minimal models, [Oprea,Tralle].

Aim.
To use the Tralle-Oprea method not to compute the cohomology of an almost abelian solvmanifold, but to find some of its properties related to formality and symplectic structures.

Commutative differential graded algebras

Let $\mathbb{K}$ ba a field of characteristic 0.

A graded $\mathbb{K}$-vector space is a family of $\mathbb{K}$-vector spaces $\mathcal{A} = \{\mathcal{A}^p\}_{p\geq 0}$. An element a of $\mathcal{A}$ has degree p, $|a| = p$, if it belongs to $\mathcal{A}^p$.

A commutative differential graded $\mathbb{K}$-algebra, cdga, $(\mathcal{A}, d)$ is a graded $\mathbb{K}$-vector space $\mathcal{A}$ together with a multiplication $\mathcal{A}^p \otimes \mathcal{A}^q \to \mathcal{A}^{p+q}$ that is

- associative,
- with unit $1 \in \mathcal{A}^0$
- such that $\forall a \in \mathcal{A}^p, b \in \mathcal{A}^q \quad a \cdot b = (-1)^{pq} b \cdot a$

and with a differential $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$ such that

- $\forall\, a \in \mathcal{A}^p, b \in \mathcal{A}^q \quad d(a \cdot b) = da \cdot b + (-1)^p a \cdot db$
- $d^2 = 0$.

$$(\mathcal{A}, d)\ \mathbb{K}\text{-cdga} \quad \Rightarrow \quad H^*(\mathcal{A}, \mathbb{K})\ \mathbb{K}\text{-cdga with } d \equiv 0.$$

A cdga $(\mathcal{M}, d)$ is minimal if

- it is free commutative, i.e. $\mathcal{M} = \bigwedge V$ with V graded vector space,
- there exist a ordered basis $\{x_\alpha\}$ of V such that
  - $V^0 = \mathbb{K}$,
  - $dV \subset \bigwedge^{\geq 2} V$
  - $dx_\alpha \in \bigwedge(x_\beta)_{\beta<\alpha}$, where with $\bigwedge^{\geq 2} V$ we mean $\bigwedge^i V$ with $i \geq 2$.

A minimal model of the cdga $(\mathcal{A}, d)$ is a minimal cdga $(\mathcal{M}, d)$ together with a cdga quasi isomorphism $\psi : \mathcal{M} \to \mathcal{A}$, i.e. a morphism that induces an isomorphism on cohomology.

Tralle-Oprea method

We can associate to every solvmanifold $S = G/\Gamma$ the Mostow fibration

$$N/\Gamma_N = (N\Gamma)/\Gamma \hookrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = \mathbb{T}^k,$$

where N is the nilradical of G.

When S is almost abelian it becomes $\mathbb{R}^n/\mathbb{Z}^n \hookrightarrow S \longrightarrow \mathbb{R}/\mathbb{Z}$.

Theorem 1 (Felix-Thomas).

[Oprea, Tralle]. Let $F \hookrightarrow E \to B$ be a fibration and let U be the largest $\pi_1(B)$-submodule of $H^*(F,\mathbb{Q})$ on which $\pi_1(B)$ acts nilpotently. Suppose that $H^*(F,\mathbb{Q})$ is a vector space of finite type and that B is a nilpotent space, then in the Sullivan model of the fibration

$$\begin{array}{ccccc}
\mathcal{A}(B) & \longrightarrow & \mathcal{A}(E) & \longrightarrow & \mathcal{A}(F) \\
\uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle \tau} & & \uparrow{\scriptstyle \rho} \\
(\bigwedge X, d_X) & \xrightarrow{\ i\ } & (\bigwedge(X \oplus Y), D) & \xrightarrow{\ q\ } & (\bigwedge Y, d_Y)
\end{array}$$

the cdga homomorphism $\rho : (\bigwedge Y, d_Y) \to \mathcal{A}(F)$ induces an isomorphism $\rho^* : H^*(\bigwedge Y, d_Y) \to U$.

For every topological space T Sullivan defined a $\mathbb{Q}$-cdga $\mathcal{A}(T)$ associated to T. Its cohomology is the cohomology of the space T over the constant sheaf $\mathbb{Q}$.
Then we can apply Theorem 1 to differential manifolds substituting $\mathbb{Q}$ with $\mathbb{R}$.

In particular by definition of Sullivan model of a fibration [Felix,Oprea,Tanré], we have that

- the diagram in Theorem 1 is commutative,
- $(\bigwedge X, d_X)$ and $(\bigwedge Y, d_Y)$ are minimal cdga,
- $\forall x \in X\ Dx = d_X x$ and $\forall y \in Y\ Dy = d_Y y + cx \wedge y'$ with $c \in \mathbb{Q}$, $x \in \bigwedge X^+$ and $y' \in \bigwedge Y^{<y}$, where with $\bigwedge X^+$ we mean all the elements in $\bigwedge X$ with degree greater than 0 and with $\bigwedge Y^{<y}$ the subalgebra of $\bigwedge Y$ generated by all the generators prior to y with respect to an order among the basis of Y,
- σ and τ are quasi isomorphisms,
- i is the inclusion and q is the projection.

In this case $\mathcal{A}(B) = \bigwedge^* \mathbb{R}$, then $(\bigwedge X, d_X) = (\bigwedge(A), 0)$ with $|A| = 1$, then for degree reasons also $(\bigwedge(X \oplus Y), D)$ is minimal and so it is the minimal model of the solvmanifold.

In general, finding U is very difficult, but when the solvmanifold is almost abelian the action is given by, [Oprea,Tralle]

$$(\bigwedge \varphi^t)^* : \mathbb{Z} \to \mathrm{Aut}(H^*(\mathbb{R}^n)).$$

To simplify the notation we denote the action $(\bigwedge \varphi^t)^*$ with φ.
By definition of nilpotent action we have that a form α is in U if and only if there exists a constant $k \in \mathbb{N}^+$ such that $(\varphi - \mathrm{Id})^k(\alpha) = 0$, where Id is the identity map.

Tralle-Oprea method:

- find U
- compute $(\bigwedge Y, d_Y)$
- compute $(\bigwedge(X \oplus Y), D)$
- compute $H^*(X \oplus Y) \cong H^*(S)$

Problem: Usually, known $(\bigwedge Y, d_Y)$, we have more than one choice for $(\bigwedge(X \oplus Y), D)$.

Formality

A minimal cdga $(\bigwedge V, d)$ is s-formal if for every $i \leq s \quad V^i = C^i \oplus N^i$ such that

- $d(C^i) = 0$
- d is injective on N^i
- $\forall n \in I_s := \bigwedge V^{\leq s} \cdot N^{\leq s}$ such that $dn = 0$, then n is exact in $\bigwedge V$.

A minimal cdga $(\bigwedge V, d)$ is formal if it is s-formal $\forall\, s$.

$$(\mathcal{M}_U, d) := (\bigwedge Y, d_Y)$$

Proposition 1 (M).

For every $\alpha, \beta \in H^*(\mathbb{R}^n)$, where $\mathbb{R}^n$ is the n-dimensional abelian Lie algebra, if α and $\beta \in U$ then also $\alpha \wedge \beta \in U$.

Proposition 2 (M).

$(\mathcal{M}_U, d)$ is always formal.

$$(\mathcal{M}_S, D) := (\bigwedge(X \oplus Y), D) = (\mathcal{M}_U \otimes \bigwedge(A), D)$$

- $DA = 0$
- $\forall x \in Y \; Dx = \left\langle \begin{array}{ll} dx & \text{or} \\ dx + x'A & \text{with } x' \in \Lambda Y^{<x} \end{array} \right.$

A generic element in $(\mathcal{M}_S, D)$ has the form $s = x + yA$ with $x, y \in \mathcal{M}_U$, then
$Ds = dx + (x' + dy)A = 0 \;\Leftrightarrow \begin{cases} dx = 0 \\ x' + dy = 0. \end{cases}$

If s is also exact, i.e. there exists $r = p + qA$ with p and $q \in \mathcal{M}_U$ such that $Dr = s$,
$\Rightarrow \begin{cases} x = dp \\ y = p' + dq \end{cases}$

A cdga $\mathcal{A}$ is of *k-finite type* if $\forall i \leq k \; \mathcal{A}^i$ is a finite dimensional vector space.

Theorem 2 (M).

If $\mathcal{M}_S$ is of k-finite type, then S is k-formal if and only if $\ker D_i|_{\mathcal{M}_U} = \ker d_i \; \forall i \leq k$, where with d_i we mean $d|_{\mathcal{M}_U^i}$.

Symplectic structures

$$S = \mathbb{R} \ltimes \mathbb{R}^{2n-1} / \mathbb{Z} \ltimes \mathbb{Z}^{2n-1}$$

A symplectic form on S is $\omega \in \bigwedge^2 S$ such that $d\omega = 0$ and $\omega^n \neq 0$.

If M is a $(2n-1)$-dimensional manifold a co-symplectic structure on M is a couple (F, η) where F is a 2-form, η is a 1-form on M, both are closed and $F^{n-1} \wedge \eta \neq 0$.

A co-symplectic structure on U is a co-symplectic structure (F, η) on $\mathbb{R}^{2n-1}$ such that $[F], [\eta] \in U$.

Proposition 3 (M).

If $\ker D_2|_{\mathcal{M}_U} = \ker d_2$ *and there exists a co-symplectic structure on* U*, then there exists a symplectic structure on* S*.*

Example

$S = \mathbb{R} \ltimes \mathbb{R}^5/\mathbb{Z} \ltimes \mathbb{Z}^5$ defined by the action of $\text{ad}_{X_6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$

with lattice generated by $t = 2\pi$.

$$\begin{aligned} H^1(S) = & \quad \langle \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle \\ H^2(S) = & \quad \langle \alpha_{16}, \alpha_{23}, \alpha_{34}, \alpha_{35}, \alpha_{45}, \alpha_{46}, \alpha_{56} \rangle \\ H^3(S) = & \quad \langle \alpha_{123}, \alpha_{126}, \alpha_{146}, \alpha_{156}, \alpha_{234}, \alpha_{235}, \alpha_{345}, \alpha_{456} \rangle \end{aligned}$$

$$\varphi = e^{2\pi \text{ad}_{X_6}} \Rightarrow \begin{aligned} \varphi(\alpha^1) = & \quad \alpha^1 + 2\pi\alpha^2 + 2\pi^2\alpha^3, \\ \varphi(\alpha^2) = & \quad \alpha^2 + 2\pi\alpha^3, \\ \varphi(\alpha^3) = & \quad \alpha^3, \\ \varphi(\alpha^4) = & \quad \alpha^4, \\ \varphi(\alpha^5) = & \quad \alpha^5. \end{aligned} \Rightarrow U \equiv H^*(\mathbb{R}^5) \Rightarrow$$

$\mathcal{M}_U \equiv \mathcal{M}_U^1 = (\bigwedge(e, f, z, p, q), 0)$,
$\mathcal{M}_S = (\bigwedge(A, e, f, z, p, q), D)$, $DA = De = Df = Dz = 0$, $Dp = eA$, $Dq = pA$.

Then for Theorem 2 S is not 1-formal.

$$F = \sum_{1\leq i<j\leq 5} a_{ij}\alpha^{ij} \quad \text{and} \quad \eta = \sum_{1\leq k\leq 5} b_k\alpha^k$$

$$\begin{aligned} \text{with } F^2\wedge\eta \neq 0 \quad \Leftrightarrow \quad & b_5(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) + \\ & + b_4(a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23}) + \\ & + b_3(a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24}) + \\ & + b_2(a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34}) + \\ & + b_1(a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}) \neq 0. \end{aligned}$$

Let $x \in \mathcal{M}_U^2$ and $y \in \mathcal{M}_U^1$ such that $\tau(x) = F$ and $\tau(y) = \eta$, then

$$x = a_{12}qp + a_{13}qe + a_{14}qf + a_{15}qz + a_{23}pe + a_{24}pf + a_{25}pz + a_{34}ef + a_{35}ez + a_{45}fz$$

and $y = b_1q + b_2p + b_3e + b_4f + b_5z$.
The element $s := x + yA \in \mathcal{M}_S^2$ is closed if and only if

$$a_{12} = a_{13} = a_{14} = a_{15} = a_{24} = a_{25} = 0.$$

Then if we consider $F = a_{23}\alpha^{23} + a_{34}\alpha^{34} + a_{35}\alpha^{35} + a_{45}\alpha^{45}$ and $\eta = \sum_{1\leq k\leq 5} b_k\alpha^k$ with $b_1 \neq 0$, $a_{23} \neq 0$, $a_{45} \neq 0$, we have a symplectic structure on S given by $\omega := F + \eta\wedge\alpha^6$.

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