Workshop
varietà reali e complesse:
geometria, topologia e analisi armonica

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SPIN(9): OCTONIONIC GEOMETRY, SPHERES AND EXCEPTIONAL SYMMETRIC SPACES

Poster by

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Spin(9) versus Sp(n)·Sp(1)

See PP12 for details

Octonions $\mathbb{O} = \mathbb{R}^8$, related with G_2, Spin(7) geometries in dimension 7, 8, appear also in higher dimensional geometries, starting with 16. In this respect the group

$$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$$

plays a special role. In the following, we present the reader with some analogies between Spin(9) structures and quaternionic Hermitian structures, which will eventually lead us to a description of the Spin(9) geometry in $\mathbb{R}^{16}$.

Analogy 1

Einstein

In the 1955 Berger list of possible holonomy groups for Riemannian manifolds, the two holonomy choices implying the Einstein curvature property are:

$\mathrm{Spin}(9)$ in $\dim = 16$ $\qquad$ $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ in $\dim = 4n$

Analogy 2

Invariant forms

The Spin(9) invariant 8-form and the quaternionic invariant 4-form are given by:

$$\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl \in \Lambda^8(\mathbb{R}^{16}) \qquad \Phi_{\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)} = \int_{\mathbb{H}P^{n-1}} p_l^* \nu_l \, dl \in \Lambda^4(\mathbb{R}^{4n})$$

[ν_l = volume form on octonionic lines $l \subset \mathbb{O}^2$, $p_l : \mathbb{O}^2 \to l$ projection, integral taken over $\mathbb{O}P^1 = S^8$ of all the $l \subset \mathbb{O}^2$, and in a similar way for $\Phi_{\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)}$].

Analogy 3

Symmetries of Hopf fibration

Spin(9) and $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ are the symmetry groups of the Hopf fibrations

$$S^{15} \longrightarrow \mathbb{O}P^1 \qquad S^{4n-1} \longrightarrow \mathbb{H}P^{n-1}$$

With all of this in mind, Friedrich's approach Fri01 for Spin(9)-structures as an analogue with (almost) quaternionic structures seems very natural. Similarly to the way a $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$-structure on M^{4n} corresponds to a subbundle $V^3 \subset \mathrm{End}(TM)$:

Definition

$\mathcal{I}_\alpha \circ \mathcal{I}_\beta$ is an almost complex structure

A Spin(9)-structure on a Riemannian manifold M^{16} is a vector subbundle $V^9 \subset \mathrm{End}(TM)$, locally spanned by $\{\mathcal{I}_1, \ldots, \mathcal{I}_9\}$ satisfying:

$$\mathcal{I}_\alpha^2 = \mathrm{Id}, \quad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha.$$

Thus, understanding the local Spin(9) geometry means understanding $V^9 \subset \mathrm{End}(\mathbb{R}^{16})$.

A simple choice

Denote by:

- R_x the octonionic right multiplication for $x \in \mathbb{O}$
- i, j, k, e, f, g, h the canonical basis of $\mathrm{Im}\,\mathbb{O}$

$$\mathcal{I}_1 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \mathcal{I}_2 = \left(\begin{array}{c|c} 0 & -R_i \\ \hline R_i & 0 \end{array}\right), \dots, \mathcal{I}_8 = \left(\begin{array}{c|c} 0 & -R_h \\ \hline R_h & 0 \end{array}\right), \mathcal{I}_9 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right).$$

Generators for V^9

Using $\mathcal{I}_1, \dots, \mathcal{I}_9$ one can explicitly describe the Spin(9) geometry of $\mathbb{R}^{16}$. In particular, one can explicitly describe the irreducible decomposition of $\Lambda^2\mathbb{R}^{16}$ under Spin(9).

Irreducible decomposition

- $\Lambda^2\mathbb{R}^{16} = \Lambda^2_{36} \oplus \Lambda^2_{84} = \mathfrak{spin}(9) \oplus \Lambda^2_{84}$
- $\mathfrak{spin}(9) = \mathrm{Span}\{\mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{1 \leq \alpha < \beta \leq 9}$
- $\Lambda^2_{84} = \mathrm{Span}\{\mathcal{I}_\alpha \circ \mathcal{I}_\beta \circ \mathcal{I}_\gamma\}_{1 \leq \alpha < \beta < \gamma \leq 9}$

Spin(9) geometry $\updownarrow$ $\mathcal{I}_1, \dots, \mathcal{I}_9$

The generators $J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta$ of $\mathfrak{spin}(9)$ can be divided into the following 3 families:

$$A = \{J_{\alpha\beta}\}_{2 \leq \alpha < \beta \leq 8}, \; B = \{J_{1\beta}\}_{2 \leq \beta \leq 8}, \; C = \{J_{\alpha 9}\}_{1 \leq \alpha \leq 8}$$

Remark 1

A, B, C give a description of the different geometries comprised in $\mathfrak{spin}(9)$:

- family A span the diagonal $\mathfrak{spin}(7)_\Delta \subset \mathfrak{spin}(9)$
- families A, B span $\mathfrak{spin}(8) \subset \mathfrak{spin}(9)$
- families A, B and C span the whole $\mathfrak{spin}(9)$

$\mathfrak{spin}(7)_\Delta \subset \mathfrak{spin}(8) \subset \mathfrak{spin}(9)$

A, B, C can be also used to recover the invariant forms of their corresponding geometries. To this aim, consider the fundamental forms $g \circ J_{\alpha\beta}$, where g is the standard metric in $\mathbb{R}^{16}$.

Remark 2

The skew-symmetric matrices

$$\psi_A = (g \circ J_{\alpha\beta})_{J_{\alpha\beta} \in A}, \quad \psi_{A,B} = (g \circ J_{\alpha\beta})_{J_{\alpha\beta} \in A \cup B}, \quad \psi_{A,B,C} = (g \circ J_{\alpha\beta})_{J_{\alpha\beta} \in A \cup B \cup C}$$

give rise to characteristic polinomials which are invariant under $\mathrm{Spin}(7)_\Delta$, Spin(8), Spin(9), respectively. The coefficients τ_i of these polynomials are differential forms of degree $2i$ in $\mathbb{R}^{16}$, which are by construction invariant for their respective geometries, so that:

- the $\mathrm{Spin}(7)_\Delta$-invariant differential form in $\mathbb{R}^{16}$ is $\tau_2(\psi_A) \in \Lambda^4(\mathbb{R}^{16})$
- the Spin(9)-invariant differential form in $\mathbb{R}^{16}$ is $\tau_4(\psi_{A,B,C}) \in \Lambda^8(\mathbb{R}^{16})$

Invariant forms

Spin(9) and spheres

See PP13b for details

The action of $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ on $\mathbb{R}^{2n}$, $\mathbb{R}^{4n}$, $\mathbb{R}^{8n}$ gives 1, 3, 7 tangent orthonormal vector fields on S^{2n-1}, S^{4n-1}, S^{8n-1}, respectively. In the following, we show how Spin(9) is responsible for the existence of more than 7 vector fields on spheres.

Classical

Maximal number $\sigma(m)$ of linearly independent vector fields on S^{m-1}:

$$\sigma(m) = 8q + 2^p - 1, \quad \text{where} \quad m = (2k+1)2^p 16^q \quad \text{and} \quad 0 \le p \le 3.$$

Only powers of 2 matter

More than 7

$m-1$	15	31	63	127	255	511	1023	2047	4095	65535
$\sigma(m)$	8	9	11	15	16	17	19	23	24	32

Some spheres S^{m-1} with more than 7 vector fields

The lowest dimensional S^{m-1} with $\sigma(m) > 7$ is $S^{15} \subset \mathbb{R}^{16}$, admitting 8 independent vector fields. A family of such 8 vector fields can be written by using $J_{19}, \dots, J_{89}$ (that is, by the discussion in the previous Section, elements of $\mathfrak{spin}(9)$ which are not in $\mathfrak{spin}(8)$). As a matter of fact, Spin(9) can be used to write down not only the 8 vector fields in S^{15}, but more generally the $8q$ vector fields appearing in $\sigma(m) = 8q + 2^p - 1$.

Spin(9) matters

It is possible to explicitly construct a maximal system of vector fields on any sphere using $J_{19}, \dots, J_{89}$ and the mentioned $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ actions.

Spin(9) gives $8q$ additional vector fields whenever $q > 0$!

We summarize this construction. In the Table, C_t and C are conjugation-like operators (their definition is omitted for simplicity), and L_x is the left multiplication for $x \in \mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

General construction

(k,p,q)	Sphere	$\sigma(m)$	Vector fields	Involved structures
$(k,0,q)$	$S^{(2k+1)16^q-1}$	$8q$	$\{\mathrm{C}_t(J_\alpha)\}_{t=1,\dots,q;\ \alpha=1,\dots,8}$	$\mathrm{Spin}(9)^q$
$(k,1,q)$	$S^{2(2k+1)16^q-1}$	$8q+1$	$\{\mathrm{C}_t(J_\alpha)\}_{t=1,\dots,q;\ \alpha=1,\dots,8}$ $\mathrm{C}(L_i)$	$\mathrm{Spin}(9)^q$ and $\mathbb{C}$
$(k,2,q)$	$S^{4(2k+1)16^q-1}$	$8q+3$	$\{\mathrm{C}_t(J_\alpha)\}_{t=1,\dots,q;\ \alpha=1,\dots,8}$ $\mathrm{C}(L_i), \mathrm{C}(L_j), \mathrm{C}(L_k)$	$\mathrm{Spin}(9)^q$ and $\mathbb{H}$
$(k,3,q)$	$S^{8(2k+1)16^q-1}$	$8q+7$	$\{\mathrm{C}_t(J_\alpha)\}_{t=1,\dots,q;\ \alpha=1,\dots,8}$ $\mathrm{C}(L_i), \dots, \mathrm{C}(L_h)$	$\mathrm{Spin}(9)^q$ and $\mathbb{O}$

More than 7? All the fault of Spin(9)

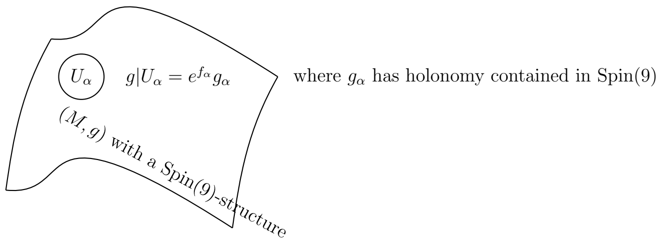
Locally conformal geometry

See OPPV for details

Parallel Spin(9) metrics are very rigid (only 3 cases can occur). What do metrics which are locally conformal to parallel Spin(9) metrics look like? In the following, we give a classification theorem in the compact case.

Definition

Locally conformally parallel Spin(9) manifold: M^{16} with a $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ structure whose induced metric g is locally conformal to metrics with $\mathrm{Hol} \subset \mathrm{Spin}(9)$.



g_α is defined on U_α

The local functions f_α give a globally defined 1-form df_α, whose g-dual B is called the Lee vector field of the locally conformally parallel structure. There is a Riemannian and totally geodesic 8-dimensional foliation $\mathcal{F}$ on M spanned by B and $\{\mathcal{I}_1 B, \ldots, \mathcal{I}_9 B\}^\perp$.

B is tangent to S^1, and $\mathcal{F}$ is given by the fibers $S^7 \times S^1$

Prototype

$S^{15} \times S^1$ is the prototype of locally conformally parallel Spin(9) manifolds. It fibers over S^8 by the octonionic Hopf fibration $S^{15} \times S^1 \to S^8$.

Local model

(1) Any locally conformally parallel Spin(9) manifold M is locally isometric to $S^{15} \times S^1$

(2) M fibers over an orbifold $\mathcal{O}^8$ finitely covered by S^8 and all fibers are finitely covered by $S^7 \times S^1$

(2) holds if the leaves of $\mathcal{F}$ are compact

Structure Theorem

M is locally conformally parallel Spin(9) if and only if the following holds:

- there is a Riemannian submersion $M \xrightarrow{\pi} S^1$
- the fibers of π are isometric to a 15-dimensional spherical space form S^{15}/K, where $K \subset \mathrm{Spin}(9)$
- the structure group of π is contained in the normalizer $N_{\mathrm{Spin}(9)}(K)$ of K in Spin(9)

Whenever M is compact

Rosenfeld planes

PP13a, work in progress

The Cayley projective plane $\mathbb{O}P^2$ usually appears as a final possibility for linear projective geometry after the three infinite series $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$. As a Riemannian manifold, one has $\mathbb{O}P^2 = \mathrm{F}_4/\mathrm{Spin}(9)$, and this leads to other "projective planes".

Rosenfeld projective planes

The symmetric spaces

$$\mathrm{E_{III}} = E_6/(\mathrm{Spin}(10)\cdot\mathrm{U}(1)), \qquad \mathrm{E_{VI}} = E_7/(\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)), \qquad \mathrm{E_{VIII}} = E_8/\mathrm{Spin}(16)$$

are referred to as the *Rosenfeld projective planes* over $\mathbb{C}\otimes\mathbb{O}$, $\mathbb{H}\otimes\mathbb{O}$, $\mathbb{O}\otimes\mathbb{O}$.

Dimension 32, 64, 128, respectively

The real cohomology of the first two is known.

Cohomology

$$H^*(\mathrm{E_{III}}) = \mathbb{R}[t,w]/(\rho_1,\rho_2), \qquad t\in H^2, w\in H^8$$
$$H^*(\mathrm{E_{VI}}) = \mathbb{R}[s,w,u]/(\sigma_1,\sigma_2,\sigma_3), \qquad s\in H^4, w\in H^8, u\in H^{12}$$

for suitable relations $\rho_1,\rho_2,\sigma_1,\sigma_2,\sigma_3$.

What are representatives for t,s,w,u?

Since $\mathrm{E_{III}}$ is Kähler and $\mathrm{E_{VI}}$ is quaternion-Kähler (a Wolf space), $t\in H^2$ and $s\in H^4$ can be represented by the Kähler 2-form of $\mathrm{E_{III}}$ and the quaternion-Kähler 4-form of $\mathrm{E_{VI}}$. In order to find representatives for w and u, we need to answer the following question.

Question

Is it possible to find families $D\subset\mathfrak{spin}(10)$ and $E\subset\mathfrak{spin}(12)$ such that Remark 1 at Page 3 is extended?

- family A spans the diagonal $\mathfrak{spin}(7)_\Delta\subset\mathfrak{spin}(9)$
- families A,B span $\mathfrak{spin}(8)\subset\mathfrak{spin}(9)$
- families A,B and C span $\mathfrak{spin}(9)$
- families A,B,C and D span $\mathfrak{spin}(10)$
- families A,B,C,D and E span $\mathfrak{spin}(12)$

$\mathrm{E_{III}}: w=\tau_4(\psi_{A,B,C,D})$
$\mathrm{E_{VI}}: w=\tau_4(\psi_{A,B,C,D,E})$
$u=\tau_6(\psi_{A,B,C,D,E})$

The case $\mathfrak{spin}(10)$

Following Bry99, consider $J_0:\mathbb{C}^{16}\to\mathbb{C}^{16}$ defined by

$$J_0 = \left(\begin{array}{c|c} i\cdot\mathrm{Id}_8 & 0 \\ \hline 0 & -i\cdot\mathrm{Id}_8\end{array}\right).$$

Then:

- $D=\{J_0,[J_0,J_{\alpha 9}]\}_{1\le\alpha\le 8}$
- the invariant forms on $\mathrm{E_{III}}$ are the Kähler form and the Spin(10)-invariant 8-form in $\mathbb{R}^{32}$ given by $\tau_4(\psi_{A,B,C,D})$

A,B,C and D span $\mathfrak{spin}(10)$

Hopefully

We hope that a similar description can be given for the quaternion-Kähler Wolf space E_{VI}, so that its Spin(12)-invariant forms in $\mathbb{R}^{64}$ are given by the quaternion-Kähler 4-form, $\tau_4(\psi_{A,B,C,D,E})$ and $\tau_6(\psi_{A,B,C,D,E})$.

Final remarks

- All of this should fit in the framework of Clifford structures as described in MS11.
- The procedure

$$\begin{array}{ccccccccc} A & \rightsquigarrow & A,B & \rightsquigarrow & A,B,C & \rightsquigarrow & A,B,C,D & \rightsquigarrow & A,B,C,D,E \\ \Phi_{\mathrm{Spin}(7)} & \rightsquigarrow & \Phi_{\mathrm{Spin}(8)} & \rightsquigarrow & \Phi_{\mathrm{Spin}(9)} & \rightsquigarrow & \Phi_{\mathrm{Spin}(10)} & \rightsquigarrow & \Phi_{\mathrm{Spin}(12)} \end{array}$$

  appears as a variation of the "Matryoshka construction" of differential forms described in DNW10.

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