SPIN(9): OCTONIONIC GEOMETRY, SPHERES AND EXCEPTIONAL SYMMETRIC SPACES

Poster by

Maurizio Parton, Università di Chieti-Pescara
parton@unich.it

Paolo Piccinni, Università di Roma “La Sapienza”
piccinni@mat.uniromal.it
Octonions $\mathbb{O} = \mathbb{R}^8$, related with $G_2$, Spin(7) geometries in dimension 7, 8, appear also in higher dimensional geometries, starting with 16. In this respect the group

$$\text{Spin}(9) \subset \text{SO}(16)$$

plays a special role. In the following, we present the reader with some analogies between Spin(9) structures and quaternionic Hermitian structures, which will eventually lead us to a description of the Spin(9) geometry in $\mathbb{R}^{16}$.

### Analogy 1

In the 1955 Berger list of possible holonomy groups for Riemannian manifolds, the two holonomy choices implying the Einstein curvature property are:

- Spin(9) in dim = 16
- Sp($n$) · Sp(1) in dim = 4n

### Analogy 2

The Spin(9) invariant 8-form and the quaternionic invariant 4-form are given by:

$$\Phi_{\text{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl \in \Lambda^8(\mathbb{R}^{16})$$

$$\Phi_{\text{Sp}(n) \cdot \text{Sp}(1)} = \int_{\mathbb{H}P^{n-1}} p_l^* \nu_l \, dl \in \Lambda^4(\mathbb{R}^{4n})$$

[$\nu_l = \text{volume form on octonionic lines } l \subset \mathbb{O}^2$, $p_l : \mathbb{O}^2 \to l$ projection, integral taken over $\mathbb{O}P^1 = S^8$ of all the $l \subset \mathbb{O}^2$, and in a similar way for $\Phi_{\text{Sp}(n) \cdot \text{Sp}(1)}$].

### Analogy 3

Spin(9) and Sp($n$) · Sp(1) are the symmetry groups of the Hopf fibrations

$$S^{15} \longrightarrow \mathbb{O}P^1$$

$$S^{4n-1} \longrightarrow \mathbb{H}P^{n-1}$$

With all of this in mind, Friedrich’s approach [FrHo1] for Spin(9)-structures as an analogue with (almost) quaternionic structures seems very natural. Similarly to the way a Sp($n$) · Sp(1)-structure on $M^{4n}$ corresponds to a subbundle $V^3 \subset \text{End}(TM)$:

### Definition

A Spin(9)-structure on a Riemannian manifold $M^{16}$ is a vector subbundle $V^9 \subset \text{End}(TM)$, locally spanned by $\{I_1, \ldots, I_9\}$ satisfying:

$$I_2^2 = \text{Id}, \quad I_\alpha^2 = I_\alpha, \quad I_\alpha \circ I_\beta = -I_\beta \circ I_\alpha.$$
Thus, understanding the local Spin(9) geometry means understanding $V^9 \subset \text{End}(\mathbb{R}^{16})$.

**A simple choice**

Denote by:

- $R_x$ the octonionic right multiplication for $x \in \mathbb{O}$
- $i, j, k, e, f, g, h$ the canonical basis of $\text{Im} \mathbb{O}$

$I_1 = \begin{pmatrix} 0 & \text{Id} & 0 \\ \text{Id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & -R_i & 0 \\ R_i & 0 & 0 \\ 0 & 0 & \text{Id} \end{pmatrix}, \ldots, I_8 = \begin{pmatrix} 0 & -R_h & 0 \\ R_h & 0 & 0 \\ 0 & 0 & \text{Id} \end{pmatrix}, I_9 = \begin{pmatrix} \text{Id} & 0 & 0 \\ 0 & 0 & -\text{Id} \\ 0 & \text{Id} & 0 \end{pmatrix}$.

Using $I_1, \ldots, I_9$ one can explicitly describe the Spin(9) geometry of $\mathbb{R}^{16}$. In particular, one can explicitly describe the irreducible decomposition of $\Lambda^2 \mathbb{R}^{16}$ under Spin(9).

**Irreducible decomposition**

- $\Lambda^2 \mathbb{R}^{16} = \Lambda_{36}^2 \oplus \Lambda_{84}^2 = \text{spin}(9) \oplus \Lambda_{84}^2$
- $\text{spin}(9) = \text{Span}\{I_\alpha \circ I_\beta\}_{1 \leq \alpha < \beta \leq 8}$
- $\Lambda_{84}^2 = \text{Span}\{I_\alpha \circ I_\beta \circ I_\gamma\}_{1 \leq \alpha < \beta < \gamma \leq 9}$

The generators $J_{\alpha\beta} = I_\alpha \circ I_\beta$ of $\text{spin}(9)$ can be divided into the following 3 families:

- $A = \{J_{\alpha\beta}\}_{2 \leq \alpha < \beta \leq 8}$
- $B = \{J_{1\beta}\}_{2 \leq \beta \leq 8}$
- $C = \{J_{\alpha 9}\}_{1 \leq \alpha \leq 8}$

**Remark 1**

$A, B, C$ give a description of the different geometries comprised in $\text{spin}(9)$:

- family $A$ span the diagonal $\text{spin}(7)_\Delta \subset \text{spin}(9)$
- families $A, B$ span $\text{spin}(8) \subset \text{spin}(9)$
- families $A, B$ and $C$ span the whole $\text{spin}(9)$

$A, B, C$ can be also used to recover the invariant forms of their corresponding geometries. To this aim, consider the fundamental forms $g \circ J_{\alpha\beta}$, where $g$ is the standard metric in $\mathbb{R}^{16}$.

**Remark 2**

The skew-symmetric matrices

$$
\psi_A = (g \circ J_{\alpha\beta})_{J_{\alpha\beta} \in A}, \quad \psi_{A,B} = (g \circ J_{\alpha\beta})_{J_{\alpha\beta} \in A \cup B}, \quad \psi_{A,B,C} = (g \circ J_{\alpha\beta})_{J_{\alpha\beta} \in A \cup B \cup C}
$$

give rise to characteristic polynomials which are invariant under Spin(7)$_\Delta$, Spin(8), Spin(9), respectively. The coefficients $\tau_i$ of these polynomials are differential forms of degree $2i$ in $\mathbb{R}^{16}$, which are by construction invariant for their respective geometries, so that:

- the Spin(7)$_\Delta$-invariant differential form in $\mathbb{R}^{16}$ is $\tau_2(\psi_A) \in \Lambda^4(\mathbb{R}^{16})$
- the Spin(9)-invariant differential form in $\mathbb{R}^{16}$ is $\tau_4(\psi_{A,B,C}) \in \Lambda^8(\mathbb{R}^{16})$
Spin(9) and spheres

The action of $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ on $\mathbb{R}^{2n}$, $\mathbb{R}^{4n}$, $\mathbb{R}^{8n}$ gives 1, 3, 7 tangent orthonormal vector fields on $S^{2n-1}$, $S^{4n-1}$, $S^{8n-1}$, respectively. In the following, we show how Spin(9) is responsible for the existence of more than 7 vector fields on spheres.

Classical

Maximal number $\sigma(m)$ of linearly independent vector fields on $S^{m-1}$:

$$\sigma(m) = 8q + 2^p - 1, \text{ where } m = (2k + 1)2^q 16^q \text{ and } 0 \leq p \leq 3.$$  

More than 7

<table>
<thead>
<tr>
<th>$m - 1$</th>
<th>15</th>
<th>31</th>
<th>63</th>
<th>127</th>
<th>255</th>
<th>511</th>
<th>1023</th>
<th>2047</th>
<th>4095</th>
<th>65535</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(m)$</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>19</td>
<td>23</td>
<td>24</td>
<td>32</td>
</tr>
</tbody>
</table>

The lowest dimensional $S^{m-1}$ with $\sigma(m) > 7$ is $S^{15} \subset \mathbb{R}^{16}$, admitting 8 independent vector fields. A family of such 8 vector fields can be written by using $J_{19}, \ldots, J_{89}$ (that is, by the discussion in the previous Section, elements of $\text{spin}(9)$ which are not in $\text{spin}(8)$). As a matter of fact, Spin(9) can be used to write down not only the 8 vector fields in $S^{15}$, but more generally the $8q$ vector fields appearing in $\sigma(m) = 8q + 2^p - 1$.

Spin(9) matters

It is possible to explicitly construct a maximal system of vector fields on any sphere using $J_{19}, \ldots, J_{89}$ and the mentioned $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ actions.

We summarize this construction. In the Table, $C_t$ and $C$ are conjugation-like operators (their definition is omitted for simplicity), and $L_x$ is the left multiplication for $x \in \mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

General construction

<table>
<thead>
<tr>
<th>$(k, p, q)$</th>
<th>Sphere</th>
<th>$\sigma(m)$</th>
<th>Vector fields</th>
<th>Involved structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(k, 0, q)$</td>
<td>$S^{2(2k+1)16^q - 1}$</td>
<td>$8q$</td>
<td>${C_t(J_a)}_{t=1, \ldots, q \atop a=1, \ldots, 8}$</td>
<td>Spin(9)$^q$</td>
</tr>
<tr>
<td>$(k, 1, q)$</td>
<td>$S^{2(2k+1)16^q - 1}$</td>
<td>$8q + 1$</td>
<td>${C_t(J_a)}_{t=1, \ldots, q \atop a=1, \ldots, 8}$</td>
<td>Spin(9)$^q$ and $\mathbb{C}$</td>
</tr>
<tr>
<td>$(k, 2, q)$</td>
<td>$S^{4(2k+1)16^q - 1}$</td>
<td>$8q + 3$</td>
<td>${C_t(J_a)}_{t=1, \ldots, q \atop a=1, \ldots, 8}$, $C(L_i), C(L_j), C(L_k)$</td>
<td>Spin(9)$^q$ and $\mathbb{H}$</td>
</tr>
<tr>
<td>$(k, 3, q)$</td>
<td>$S^{8(2k+1)16^q - 1}$</td>
<td>$8q + 7$</td>
<td>${C_t(J_a)}_{t=1, \ldots, q \atop a=1, \ldots, 8}$, $C(L_i), \ldots, C(L_h)$</td>
<td>Spin(9)$^q$ and $\mathbb{O}$</td>
</tr>
</tbody>
</table>
Parallel Spin(9) metrics are very rigid (only 3 cases can occur). What do metrics which are locally conformal to parallel Spin(9) metrics look like? In the following, we give a classification theorem in the compact case.

**Definition**

Locally conformally parallel Spin(9) manifold: $M^{16}$ with a Spin(9) $\subset SO(16)$ structure whose induced metric $g$ is locally conformal to metrics with Hol $\subset$ Spin(9).

$\left( M, g \right)$ with a Spin(9)-structure $\\Gamma^{\alpha} \ w.r.t.\ g$$

where $g_{\alpha}$ has holonomy contained in Spin(9)

The local functions $f_{\alpha}$ give a globally defined 1-form $df_{\alpha}$, whose $g$-dual $B$ is called the Lee vector field of the locally conformally parallel structure. There is a Riemannian and totally geodesic 8-dimensional foliation $F$ on $M$ spanned by $B$ and $\{I_1 B, \ldots, I_9 B\}^\perp$.

**Prototype**

$S^{15} \times S^1$ is the prototype of locally conformally parallel Spin(9) manifolds. It fibers over $S^8$ by the octonionic Hopf fibration $S^{15} \times S^1 \to S^8$.

**Local model**

1. Any locally conformally parallel Spin(9) manifold $M$ is locally isometric to $S^{15} \times S^1$.
2. $M$ fibers over an orbifold $O^8$ finitely covered by $S^8$ and all fibers are finitely covered by $S^7 \times S^1$.

**Structure Theorem**

$M$ is locally conformally parallel Spin(9) if and only if the following holds:

- there is a Riemannian submersion $M \xrightarrow{\pi} S^1$
- the fibers of $\pi$ are isometric to a 15-dimensional spherical space form $S^{15}/K$, where $K \subset$ Spin(9)
- the structure group of $\pi$ is contained in the normalizer $N_{\text{Spin}(9)}(K)$ of $K$ in Spin(9)
The Cayley projective plane $\mathbb{O}P^2$ usually appears as a final possibility for linear projective geometry after the three infinite series $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$. As a Riemannian manifold, one has $\mathbb{O}P^2 = F_4/\text{Spin}(9)$, and this leads to other “projective planes”.

The symmetric spaces $E_{\text{III}} = E_6/(\text{Spin}(10)\cdot\text{U}(1))$, $E_{\text{VI}} = E_7/(\text{Spin}(12)\cdot\text{Sp}(1))$, $E_{\text{VIII}} = E_8/\text{Spin}(16)$ are referred to as the Rosenfeld projective planes over $\mathbb{C} \otimes \mathbb{O}$, $\mathbb{H} \otimes \mathbb{O}$, $\mathbb{O} \otimes \mathbb{O}$.

The real cohomology of the first two is known.

\[ H^*(E_{\text{III}}) = \mathbb{R}[t, w]/(\rho_1, \rho_2), \quad t \in H^2, w \in H^8 \]
\[ H^*(E_{\text{VI}}) = \mathbb{R}[s, w, u]/(\sigma_1, \sigma_2, \sigma_3), \quad s \in H^4, w \in H^8, u \in H^{12} \]
for suitable relations $\rho_1, \rho_2, \sigma_1, \sigma_2, \sigma_3$.

Since $E_{\text{III}}$ is Kähler and $E_{\text{VI}}$ is quaternion-Kähler (a Wolf space), $t \in H^2$ and $s \in H^4$ can be represented by the Kähler 2-form of $E_{\text{III}}$ and the quaternion-Kähler 4-form of $E_{\text{VI}}$. In order to find representatives for $w$ and $u$, we need to answer the following question.

**Question**

Is it possible to find families $D \subset \text{spin}(10)$ and $E \subset \text{spin}(12)$ such that Remark 1 at Page 3 is extended?

- family $A$ spans the diagonal $\text{spin}(7)_{\Delta} \subset \text{spin}(9)$
- families $A, B$ span $\text{spin}(8) \subset \text{spin}(9)$
- families $A, B$ and $C$ span $\text{spin}(9)$
- families $A, B, C$ and $D$ span $\text{spin}(10)$
- families $A, B, C, D$ and $E$ span $\text{spin}(12)$

**The case spin(10)**

Following [Bry99] consider $J_0 : \mathbb{C}^{16} \to \mathbb{C}^{16}$ defined by

\[ J_0 = \begin{pmatrix} \frac{i \cdot \text{Id}_8}{0} & 0 \\ 0 & \frac{-i \cdot \text{Id}_8}{0} \end{pmatrix}. \]

Then:

- $D = \{ J_0, [J_0, J_{\alpha}] \}_{1 \leq \alpha \leq 8}$
- the invariant forms on $E_{\text{III}}$ are the Kähler form and the Spin(10)-invariant 8-form in $\mathbb{R}^{32}$ given by $\tau_4(\psi_{A,B,C,D})$
We hope that a similar description can be given for the quaternion-Kähler Wolf space $E_{VI}$, so that its Spin(12)-invariant forms in $\mathbb{R}^{64}$ are given by the quaternion-Kähler 4-form, $\tau_4(\psi_{A,B,C,D,E})$ and $\tau_6(\psi_{A,B,C,D,E})$.

Hopefully

All of this should fit in the framework of Clifford structures as described in [MS11].

The procedure

\[
\begin{align*}
A & \rightsquigarrow A, B \\
& \rightsquigarrow A, B, C \\
& \rightsquigarrow A, B, C, D \\
& \rightsquigarrow A, B, C, D, E
\end{align*}
\]

\[
\Phi_{\text{Spin}(7)} \rightsquigarrow \Phi_{\text{Spin}(8)} \rightsquigarrow \Phi_{\text{Spin}(9)} \rightsquigarrow \Phi_{\text{Spin}(10)} \rightsquigarrow \Phi_{\text{Spin}(12)}
\]

appears as a variation of the “Matryoshka construction” of differential forms described in [DNW10].

Final remarks

- All of this should fit in the framework of Clifford structures as described in [MS11].
- The procedure

\[
\begin{align*}
A & \rightsquigarrow A, B \\
& \rightsquigarrow A, B, C \\
& \rightsquigarrow A, B, C, D \\
& \rightsquigarrow A, B, C, D, E
\end{align*}
\]

\[
\Phi_{\text{Spin}(7)} \rightsquigarrow \Phi_{\text{Spin}(8)} \rightsquigarrow \Phi_{\text{Spin}(9)} \rightsquigarrow \Phi_{\text{Spin}(10)} \rightsquigarrow \Phi_{\text{Spin}(12)}
\]

appears as a variation of the “Matryoshka construction” of differential forms described in [DNW10].

References

[Bae] J. C. Baez. This Week’s Finds in Mathematical Physics. Weeks 64 and 106.


