A parabolic flow of balanced metrics

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BALANCED METRICS

Balanced metrics were introduced by Michelson in

On the existence of special metrics in complex geometry, Acta Math. **149** (1982).

A Hermitian metric on an n-dimensional complex manifold is called *Balanced* if its fundamental form ω satisfies

$$d^*\omega = 0$$

or, equivalently,

$$d\omega^{n-1} = 0\,.$$

The balanced condition is more general than the Kähler one

$$d\omega = 0\,.$$

Some motivations for studying balanced structures

- The existence of a balanced metric can be characterized in terms of currents. (Michelson).
- A modification of a balanced manifold is still balanced (Alessandrini-Bassanelli).
- The twistor space of an anti-self-dual 4-Riemannian manifold is always a balanced manifold (Michelson).
- Tori bundles over Riemannian surfaces provide examples of balanced manifolds (Gray).
- The connected sum of copies of $S^3 \times S^3$ admits a balanced metrics (Fu-Li-Yau)
- Nilmanifolds (or more general solvmanifolds) provide examples of balanced manifolds.

The Calabi flow is a 4-order flow on Kähler manifolds whose study is strictly related to the following

Calabi's Problem. Given a compact Kähler manifold (M, J, ω_0)*, find a Kähler form in* $[\omega_0] \in H^2(M, \mathbb{C})$ *having minimal scalar curvature.*

The Calabi functional is then defined as $Ca(\omega) = \int_M s_\omega^2 V_\omega$, where s_ω is the scalar curvature of ω and V_ω is the volume form.

The *Calabi flow* is the "gradient flow"of Ca which is defined as

$$\frac{d}{dt}\omega = i\partial\bar{\partial}s_\omega \,, \quad \omega(0) = \omega_0 \,.$$

E. Calabi, *Extremal Kähler metric*, in Annals of Mathematics Studies (1982).

Solutions to the Calabi flow

Theorem [Chen, He]. *The Calabi flow has always a unique short-time solution existing as long as the Ricci curvature stays uniformly bounded. Moreover, Kälher metrics having constant scalar curvature are stable.*

X.X. Chen, W.Y. He, *On the Calabi flow*, Amer. J. Math. (2008).

The Calabi flow can be rewritten in terms of $\varphi = \omega^{n-1}$ as

$$\frac{d}{dt}\varphi = i\partial\bar{\partial} * (\rho \wedge *\varphi)\,, \quad \varphi(0) = (\omega_0)^{n-1}\,,$$

where ρ is the Ricci form. This other flow still makes sense in the balanced context but it is not still elliptic.

We focus on the problem: *Generalizing the Calabi flow to the context of balanced metrics.*

In the landscape of balanced Geometry the so-called *Bott-Chern cohomology complex* has a special role. Such a complex is defined as

$$H_{BC}(M) = \frac{\ker d}{\mathrm{Im}(\partial\overline{\partial})},$$

and it can be realized has the kernel of the "Laplacian" operator

$$\Delta_{BC} = \partial\overline{\partial}\overline{\partial}^*\partial^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \overline{\partial}^*\partial\partial^*\overline{\partial} + \partial^*\overline{\partial}\overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} + \partial^*\partial\,.$$

We consider the *generalized Calabi flow*

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = i\partial\bar{\partial} *_t (\rho_t \wedge *_t\varphi(t)) - (n-1)\Delta_{BC}\varphi(t) \\ d\varphi(t) = 0 \\ \varphi(0) = (\omega_0)^{n-1}\,. \end{cases}$$

Our main results is about the existence of a solution to the generalize Calabi flow:

Theorem.[Bedulli, V.] *The generalized Calabi flow*

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = i\partial\overline{\partial} *_t (\rho_t \wedge *_t\varphi(t)) - (n-1)\Delta_{BC}\varphi(t) \\ d\varphi(t) = 0 \\ \varphi(0) = (\omega_0)^{n-1} . \end{cases}$$

admits a unique solution in the Bott-Chern class of $[(\omega_0)^{n-1}]$ *defined in a maximal interval* $[0, \epsilon)$. *Moreover if the initial structure is Kähler then the flow reduces to the classical Calabi flow.*

An explicit example

The *Iwasawa manifold* is the compact complex manifold defined as the quotient

$$M = \mathrm{H}_3(\mathbb{C})/\Gamma$$

where $\mathrm{H}_3(\mathbb{C})$ is the 3-dimensional *complex Heisenberg group* and Γ is a co-compact lattice.

M has a global $(1,0)$-coframe $\{\alpha^1, \alpha^2, \alpha^3\}$ satisfying

$$d\alpha^1 = d\alpha^2 = 0\,, \quad d\alpha^3 = \alpha^1 \wedge \alpha^2\,.$$

The generalized Calabi flow with initial condition

$$\omega_0 = i\alpha^1 \wedge \alpha^{\bar{1}} + i\alpha^2 \wedge \alpha^{\bar{2}} + i\alpha^3 \wedge \alpha^{\bar{3}}$$

has solution

$$\omega(t) = i\sqrt{1-4t}\,\alpha^1 \wedge \alpha^{\bar{1}} + i\sqrt{1-4t}\,\alpha^2 \wedge \alpha^{\bar{2}} + i\frac{1}{\sqrt{1-4t}}\,\alpha^3 \wedge \alpha^{\bar{3}}.$$